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Dust in the York Canonical Basis of ADM Tetrad Gravity: the Problem of Vorticity.

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Abstract

Brown's formulation of dynamical perfect fluids in Minkowski space-time is extended to ADM tetrad gravity in globally hyperbolic, asymptotically Minkowskian space-times. For the dust we get the Hamiltonian description in closed form in the York canonical basis, where we can separate the inertial gauge variables of the gravitational field in the non-Euclidean 3-spaces of global non-inertial frames from the physical tidal ones. After writing the Hamilton equations of the dust, we identify the sector of irrotational motions and the gauge fixings forcing the dust 3-spaces to coincide with the 3-spaces of the non-inertial frame. The role of the inertial gauge variable York time (the remnant of the clock synchronization gauge freedom) is emphasized. Finally the Hamiltonian Post-Minkowskian linearization is studied. The future application of this formalism will be the study of cosmological back-reaction (as an alternative to dark energy) in the York canonical basis.

I. INTRODUCTION

By using Dirac's theory of constraints in a series of papers [1–4] we developed the Hamiltonian formulation of ADM tetrad gravity in suitable globally hyperbolic, asymptotically Minkowskian space-times admitting global 3+1 splittings (i.e. global non-inertial frames) with dynamical matter represented by positive-energy charged scalar point particles and by the electro-magnetic field in the radiation gauge. By means of a Shanmugadhasan canonical transformation, implementing the York map [5], we found a York canonical basis adapted to ten of the fourteen first class constraints of tetrad gravity and we could identify the gauge variables (the *inertial effects*) and the physical degrees of freedom (the *tidal effects*) of the gravitational field in global non-inertial frames (the only ones allowed by the equivalence principle).

In Refs.[3, 4] we also developed the Hamiltonian Post-Minkowskian (HPM) linearization in a family of (non-harmonic) 3-orthogonal Schwinger time gauges. In particular it turns out [4] that at least part of the astrophysical and cosmological *dark matter* could arise as a *relativistic inertial effect* determined by the trace of the extrinsic curvature (York time) of the instantaneous 3-spaces as 3-manifolds embedded into space-time. This inertial gauge variable is the general relativistic remnant of the special relativistic gauge freedom in clock synchronization as a definition of instantaneous 3-space. In canonical general relativity the 3-spaces are dynamically determined modulo this gauge freedom [6].

With the exception of the three gauge variables describing the freedom in the choice of the 3-coordinates inside the instantaneous 3-spaces of the non-inertial frame all the other inertial and tidal effects are 3-scalar fields on the 3-spaces in this Hamiltonian formulation. The 3-scalar nature of the tidal variables and of many of the inertial ones opens the possibility to study the averaging over 3-volumes of most of the Hamilton equations (and not of only a small subset as in the standard approaches), extending the results of Buchert approach [7] to cosmology in which the *back-reaction* arising from the volume-average of the non-linear Einstein equations is proposed as an alternative to *dark energy.* It will be interesting to see which is the role of the York time in a framework using neither 3-spaces comoving with an irrotational fluid nor comoving coordinates (see Ref.[8] for the reformulation of the approach of Ref.[7] in arbitrary coordinate systems).

But before doing this attempt we need to include dynamical perfect fluids in our framework, because this is the standard matter (very often only test matter) in cosmology. Moreover we want to avoid the use of irrotational test fluids in cosmological comoving coordinates, because this simplification does not allow to make explicit the gauge freedom in the York time.

In this paper we define the general relativistic extension of Brown's approach [9] to dynamical perfect fluids in Minkowski space-time. Our starting point will be Refs.[10, 11], where we reformulated this approach in the framework of the inertial rest frames, a subcase of the general treatment of special relativistic non-inertial frames developed in Ref.[12].

We consider in detail dust, because, as shown in Refs.[10], only in this case (and in few others including the photon gas) we can get a closed form for the Hamiltonian mass density of the fluid [1].

[1] In general the transition from velocities to momenta in this approach leads to a trascendental equation,

In Section II we give a review of the 3+1 point of view which allows to define global non-inertial frames with well defined instantaneous 3-spaces necessary for the formulation of the Cauchy problem of field equations. After the special relativistic formulation based on Ref.[12], we give its extension to the class of space-times used in ADM tetrad gravity [1, 2] together with the definitions of the tetrad and cotetrad fields. In both cases there are two congruences of time-like observers associated with each 3+1 splitting of space-time: i) the congruence of the Eulerian observers, whose 4-velocity is the unit normal to the 3-spaces; ii) the skew, in general non surface-forming, congruence, whose 4-velocity is the evolution vector field associated with the 3+1 splitting of space-time.

After having clarified this kinematical scenario we review in Section III Brown's approach [9] to perfect fluids in Minkowski space-time in the formulation of Ref.[10] employing the rest-frame instant form of the dynamics of isolated systems [12]. We show that we can get the explicit Hamiltonian expression of the energy-momentum tensor and of the unit 4-velocity of the fluid only for special equations of state, in particular for dust. In the rest of the paper only the dust will be considered. The dust is not irrotational so that its 4-velocity is not surface-forming.

Then in Section IV we study the action of ADM tetrad gravity coupled to dynamical matter used in Refs.[1, 2] with the matter action of a dust. After having obtained the Hamiltonian formulation we make the canonical transformation to the York canonical basis with the resulting expressions for the 4-velocity, the energy-momentum tensor and the dust Hamilton equations. Also the expressions in the family of (non-harmonic) 3-orthogonal Schwinger time gauges, where the spatial metric is diagonal, is given. Finally we present the Hamiltonian Post-Minkowskian linearization of Refs.[3, 4] in the case of dust.

Section V contains a study of the skew congruence of time-like observers defined by the unit 4-velocity of the dust (their world-lines are the flux lines). It is shown that a gauge fixing of the inertial shift functions of the gravitational field (it is a restriction on the 3-coordinates on the 3-spaces) is needed to make it to coincide with the skew congruence of the 3+1 splitting of space-time. After analyzing the properties of the dust flux lines, we introduce the Eulerian point of view for the description of the dust following Ref.[11]. Then we introduce the acceleration, the expansion, the shear and the vorticity of the dust. We find which are the two first class Hamiltonian constraints to be added by hand to restrict the dust motions to the irrotational ones. These motions are described only by one pair of canonical variables and their 4-velocity is surface-forming: it can be made to coincide with the unit normal to the 3-spaces of the 3+1 splitting by adding a gauge fixing on the inertial lapse function (it is a restriction on the York time, i.e. on the clock synchronization freedom). In this gauge the irrotational dust is comoving with the 3-space and the flux lines are the world-lines of the Eulerian observers. We studied all these properties in detail, because usually these topics are not discussed in the literature.

In the Conclusions we make some comments on the future use of these result in cosmology.

In Appendix A there is a review of the York canonical basis of ADM tetrad gravity [1, 2], followed by the definition of the acceleration, expansion and shear of the irrotational Eulerian observers whose 4-velocity is the unit normal to the instantaneous 3-spaces.

whose explicit solution is known in few cases.

In Appendix B there the discussion on the relation between Lagrangian velocity and Hamiltonian momenta as a function of the equation of state of the perfect fluid.

II. THE 3+1 POINT OF VIEW AND GLOBAL NON-INERTIAL FRAMES IN SPECIAL AND GENERAL RELATIVITY

In this Section we review some aspects of the theory of global non-inertial frames, the only ones admitted in general relativity due to the equivalence principle. Their formulation in Minkowski space-time was given in Ref.[12]. Then it was extended to globally hyperbolic, asymptotically Minkowskian Einstein space-times without super-translations in Refs.[1, 2].

A. Minkowski Space-Time

As shown in Ref.[12] we now have a metrology-oriented description of non-inertial frames in special relativity. This can be done with the *3+1 point of view* and the use of observer-dependent Lorentz-scalar radar 4-coordinates. Let us give the world-line $x^\mu(\tau)$ of an arbitrary time-like observer carrying a standard atomic clock: τ is an arbitrary monotonically increasing function of the proper time of this clock. Then we give an admissible 3+1 splitting of Minkowski space-time, namely a nice foliation with space-like instantaneous 3-spaces Σ_τ: it is the mathematical idealization of a protocol for clock synchronization (all the clocks in the points of Σ_τ sign the same time of the atomic clock of the observer). On each 3-space Σ_τ we choose curvilinear 3-coordinates σ^r having the observer as origin. These are the *radar 4-coordinates* $\sigma^A = (\tau; \sigma^r)$. If $x^\mu \mapsto \sigma^A(x)$ is the coordinate transformation from the Cartesian 4-coordinates x^μ of a reference inertial observer to radar coordinates, its inverse $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$ defines the *embedding* functions $z^\mu(\tau, \sigma^r)$ describing the 3-spaces Σ_τ as embedded 3-manifold into Minkowski space-time. From now on we shall denote the curvilinear 3-coordinates σ^r with the notation $\vec{\sigma}$ for the sake of simplicity.

The induced 4-metric on Σ_τ is the following functional of the embedding: ${}^4g_{AB}(\tau, \vec{\sigma}) = [z^\mu_A\, \eta_{\mu\nu}\, z^\nu_B](\tau, \vec{\sigma})$, where $z^\mu_A = \partial\, z^\mu / \partial\, \sigma^A$ and ${}^4\eta_{\mu\nu}$ is the flat metric. The 4-metric ${}^4g_{AB}$ has signature $\epsilon\, (+---)$ with $\epsilon = \pm$ (the particle physics, $\epsilon = +$, and general relativity, $\epsilon = -$, conventions); the flat Minkowski metric is $\eta_{\mu\nu} = \epsilon\, (+---)$.

While the 4-vectors $z^\mu_r(\tau, \vec{\sigma})$ are tangent to Σ_τ, so that the unit normal $l^\mu(\tau, \vec{\sigma})$ is proportional to $\epsilon^\mu{}_{\alpha\beta\gamma}\, [z^\alpha_1\, z^\beta_2\, z^\gamma_3](\tau, \vec{\sigma})$, we have $z^\mu_\tau(\tau, \vec{\sigma}) = [N\, l^\mu + n^r\, z^\mu_r](\tau, \vec{\sigma})$ for the so-called evolution 4-vector, where $N(\tau, \vec{\sigma}) = 1 + n(\tau, \vec{\sigma}) = \epsilon\, [z^\mu_\tau\, l_\mu](\tau, \vec{\sigma})$ and $n_r(\tau, \vec{\sigma}) = -\epsilon\, g_{\tau r}(\tau, \vec{\sigma}) = [{}^3g_{rs}\, n^s](\tau, \vec{\sigma})$ are the lapse and shift functions. We also have $|det\, {}^4g| = (1 + n)\, \sqrt{\gamma}$; $\sqrt{\gamma} = \sqrt{det\, {}^3g}$ with ${}^3g_{rs} = -\epsilon\, {}^4g_{rs}$ of positive signature.

The conditions for having an admissible 3+1 splitting of space-time are:
a) $1 + n(\tau, \vec{\sigma}) > 0$ everywhere (the instantaneous 3-spaces never intersect each other, so that there are no coordinate singularities like it happen with Fermi coordinates);
b) the Møller conditions [12], which imply
i) $\epsilon\, {}^4g_{\tau\tau} > 0$, i.e. $(1 + n)^2 > \sum_r n_r\, n^r$ (the rotational velocity never exceeds the velocity of light c, so that there no coordinate singularities like it happens with the rotating disk);
ii) $\epsilon\, {}^4g_{rr} = -{}^3g_{rr} < 0$ (satisfied by the signature of ${}^3g_{rs}$), ${}^4g_{rr}\, {}^4g_{ss} - ({}^4g_{rs})^2 > 0$ and $det\, \epsilon\, {}^4g_{rs} = -det\, {}^3g_{rs} < 0$ (satisfied by the signature of ${}^3g_{rs}$) so that $det\, {}^4g_{AB} < 0$ (these conditions imply that ${}^3g_{rs}$ has three definite positive eigenvalues $\lambda_r = \Lambda_r^2$ in the non-degenerate case without Killing symmetries, the only one we consider).

In this 3+1 point of view the embedding functions $z^\mu(\tau, \vec{\sigma})$ describe the inertial effects present in the given non-inertial frame. The 4-metric ${}^4g_{AB}(\tau, \vec{\sigma})$ is the potential for the induced inertial effects. For instance the extrinsic curvature ${}^3K_{rs}(\tau, \vec{\sigma}) = \Big(\frac{1}{2\,(1+n)}\,(n_{r|s} + n_{s|r} - \partial_\tau\,{}^3g_{rs})\Big)(\tau, \vec{\sigma})$ of the non-Euclidean 3-spaces Σ_τ ($|r$ denotes the covariant derivative in it) is one of these induced inertial effects. All these inertial effects derive from the gauge freedoms of clock synchronization and choice of the 3-coordinates.

In Ref.[12] there is a complete description of isolated systems in non-inertial frames by means of *parametrized Minkowski theories* where there is a well defined action principle containing the embeddings $z^\mu(\tau, \vec{\sigma})$ as Lagrangian variables and implying their gauge nature.

In special relativity we can restrict ourselves to inertial frames and define the *inertial rest-frame instant form of dynamics for isolated systems* by choosing the 3+1 splitting corresponding to the intrinsic inertial rest frame of the isolated system centered on an inertial observer: the instantaneous 3-spaces, named Wigner 3-space due to the fact that the 3-vectors inside them are Wigner spin-1 3-vectors [12], are orthogonal to the conserved 4-momentum P^μ of the configuration. In Ref.[12] there is the extension to admissible *non-inertial rest frames*, where P^μ is orthogonal to the asymptotic space-like hyper-planes to which the instantaneous 3-spaces tend at spatial infinity. This non-inertial family of 3+1 splittings is the only one admitted by the asymptotically Minkowskian space-times covered by the canonical gravity formulation of Refs.[1–4].

B. Globally Hyperbolic, Asymptotically Minkowskian Einstein Space-times

In the globally hyperbolic, asymptotically Minkowskian space-times without super-translations of Ref.[1] we can use radar 4-coordinates $\sigma^A = (\sigma^\tau = \tau; \sigma^r)$, $A = \tau, r$, adapted to an admissible 3+1 splitting of the space-time and centered on an arbitrary time-like observer like in special relativity: they define a non-inertial frame centered on the observer. The absence of super-translations implies [13] that the instantaneous 3-spaces Σ_τ are orthogonal to the conserved ADM 4-momentum at spatial infinity, i.e. they are non-inertial rest frames of the 3-universe. Therefore at spatial infinity there is an *asymptotic Minkowski background 4-metric* and there are asymptotic inertial observers to be identified with the fixed stars of the star catalogues.

The 3-spaces Σ_τ are identified by the embeddings $x^\mu = z^\mu(\tau, \vec{\sigma})$, but now the quantities $z^\mu_A(\tau, \vec{\sigma}) = \frac{\partial\, z^\mu(\tau, \vec{\sigma})}{\partial\, \sigma^A}$ are the transition coefficients from the adapted radar 4-coordinates $\sigma^A = (\tau, \sigma^r)$ to world 4-coordinates x^μ (${}^4g_{AB} = z^\mu_A\, {}^4g_{\mu\nu}\, z^\nu_B$). As in special relativity the space-like 4-vectors $z^\mu_r(\tau, \vec{\sigma})$ are tangent to the 3-spaces, the unit normal to them is $l^\mu(\tau, \vec{\sigma}) = \Big(z^\mu_A\, l^A\Big)(\tau, \vec{\sigma}) = \Big(\frac{{}^4g^{\mu\nu}\, \sqrt{|det\, {}^4g_{\rho\sigma}|}}{\tilde{\phi}}\, \epsilon_{\mu\alpha\beta\gamma}\, z^\alpha_1\, z^\beta_2\, z^\gamma_3\Big)(\tau, \vec{\sigma})$ and the time-like evolution 4-vector $z^\mu_\tau(\tau, \vec{\sigma})$ has the decomposition $z^\mu_\tau(\tau, \vec{\sigma}) = \Big((1 + n)\, l^\mu + n^r\, z^\mu_r\Big)(\tau, \vec{\sigma})$, with $v^\mu(\tau, \vec{\sigma}) = \Big(z^\mu_A\, v^A\Big)(\tau, \vec{\sigma}) = \Big(z^\mu_\tau / \sqrt{\epsilon\, {}^4g_{\mu\nu}\, z^\mu_\tau\, z^\nu_\tau}\Big)(\tau, \vec{\sigma})$ the associated unit time-like 4-vector (skew with respect to the foliation with 3-spaces).

In general relativity, where the 3-spaces are dynamically determined [6] except for the York time ${}^3K$ (the general relativistic remnant of the special relativistic gauge freedom in clock synchronization; differently from special relativity ${}^3K(\tau, \vec{\sigma})$ is an independent inertial gauge variable and not an induced inertial effect), it is convenient to use the embedding

$z^\mu(\tau,\vec{\sigma}) = x^\mu_o + \epsilon^\mu_\tau\,\tau + \epsilon^\mu_r\,\sigma^r$, where ϵ^μ_A are asymptotic flat tetrads (ϵ^μ_τ is orthogonal to the asymptotic flat Euclidean 3-space, being proportional to the conserved ADM 4-momentum).

In Refs. [1–4](see also Appendix A) we introduced ADM tetrad gravity by considering the ADM action as a functional of cotetrads ${}^4E^{(\alpha)}_A(\tau,\vec{\sigma}) = z^\mu_A(\tau,\vec{\sigma})\,{}^4E^{(\alpha)}_\mu(\tau,\vec{\sigma})$ [2] by using the following decomposition of the 4-metric in radar 4-coordinates

$$
\begin{aligned}
{}^4g_{AB} &= {}^4E^{(\alpha)}_A\,{}^4\eta_{(\alpha)(\beta)}\,{}^4E^{(\beta)}_B = {}^4\overset{\circ}{E}{}^{(\alpha)}_A\,{}^4\eta_{(\alpha)(\beta)}\,{}^4\overset{\circ}{E}{}^{(\beta)}_B\,,\\
{}^4E^{(\alpha)}_A &= L^{(\alpha)}{}_{(\beta)}(\varphi_{(c)})\,{}^4\overset{\circ}{E}{}^{(\beta)}_A = L^{(\alpha)}{}_{(o)}(\varphi_{(c)})\,{}^4\overset{\circ}{\bar{E}}{}^{(o)}_A + L^{(\alpha)}_{(a)}(\varphi_{(c)})\,R^T_{(a)(b)}(\alpha_{(d)})\,{}^4\overset{\circ}{\bar{E}}{}^{(b)}_A\,.
\end{aligned}
\tag{2.1}
$$

The cotetrads ${}^4\overset{\circ}{E}{}^{(\alpha)}_A$ are adapted to the 3+1 splitting (the time-like adapted tetrad is the unit normal l^A to Σ_τ) by a point-dependent standard Lorentz boost for time-like orbits acting on the flat indices parametrized by $\varphi_{(a)}(\tau,\vec{\sigma})$ [3]. The adapted tetrads and cotetrads (corresponding to the so called *Schwinger time gauges*) have the expression

$$
\begin{aligned}
{}^4\overset{\circ}{E}{}^A_{(o)} &= \frac{1}{1+n}\,(1; -n_{(a)}\,{}^3e^r_{(a)}) = l^A, \qquad {}^4\overset{\circ}{E}{}^A_{(a)} = (0; {}^3e^r_{(a)}),\\
{}^4\overset{\circ}{E}{}^{(o)}_A &= (1+n)\,(1;\vec{0}) = \epsilon\, l_A, \qquad {}^4\overset{\circ}{E}{}^{(a)}_A = (n_{(a)}; {}^3e_{(a)r}),
\end{aligned}
\tag{2.2}
$$

where ${}^3e^r_{(a)}(\tau,\vec{\sigma})$ and ${}^3e_{(a)r}(\tau,\vec{\sigma})$ are triads and cotriads on Σ_τ and $n_{(a)}(\tau,\vec{\sigma}) = \Big(n_r\,{}^3e^r_{(a)}\Big)(\tau,\vec{\sigma}) = \Big(n^r\,{}^3e_{(a)r}\Big)(\tau,\vec{\sigma})$ [4] are adapted shift functions vanishing at spatial infinity.

The adapted tetrads ${}^4\overset{\circ}{E}{}^A_{(a)}$ and triads ${}^3e^r_{(a)}$ are defined modulo SO(3) rotations ${}^4\overset{\circ}{E}{}^A_{(a)} = R_{(a)(b)}(\alpha_{(e)})\,{}^4\overset{\circ}{\bar{E}}{}^A_{(b)}$, ${}^3e^r_{(a)} = R_{(a)(b)}(\alpha_{(e)})\,{}^3\bar{e}^r_{(b)}$, where $\alpha_{(a)}(\tau,\vec{\sigma})$ are three point-dependent Euler angles. After having chosen an arbitrary point-dependent origin $\alpha_{(a)}(\tau,\vec{\sigma}) = 0$, we arrive at the following adapted tetrads and cotetrads [$\bar{n}_{(a)} = \sum_b n_{(b)}\,R_{(b)(a)}(\alpha_{(e)})$]

---

[2] Flat indices (α), $\alpha = o, a$, are raised and lowered by the flat Minkowski metric ${}^4\eta_{(\alpha)(\beta)} = \epsilon\,(+---)$. We define ${}^4\eta_{(a)(b)} = -\epsilon\,\delta_{(a)(b)}$ with a positive-definite Euclidean 3-metric.

[3] As shown in Ref.[1, 13], the flat indices (a) of the adapted tetrads and cotetrads and of the triads and cotriads on Σ_τ transform as Wigner spin-1 indices under the point-dependent SO(3) Wigner rotations associated with Lorentz transformations $\Lambda^{(\alpha)}{}_{(\beta)}(z)$ in the tangent plane to the space-time in the given point of Σ_τ. Instead the index (o) of the adapted tetrads and cotetrads is a local Lorentz scalar index.

[4] Since we use the positive-definite 3-metric $\delta_{(a)(b)}$, we shall use only lower flat spatial indices. Therefore for the cotriads we use the notation ${}^3e^{(a)}_r \overset{def}{=} {}^3e_{(a)r}$ with $\delta_{(a)(b)} = {}^3e^r_{(a)}\,{}^3e_{(b)r}$.

$$
\begin{aligned}
{}^4\bar{\overset{\circ}{E}}{}^A_{(o)} &= {}^4\overset{\circ}{E}{}^A_{(o)} = \frac{1}{1+n}\,(1; -\bar{n}_{(a)}\,{}^3\bar{e}^r_{(a)}) = l^A, \qquad {}^4\bar{\overset{\circ}{E}}{}^A_{(a)} = (0; {}^3\bar{e}^r_{(a)}),\\
{}^4\bar{\overset{\circ}{E}}{}^{(o)}_A &= {}^4\overset{\circ}{E}{}^{(o)}_A = (1+n)\,(1; \vec{0}) = \epsilon\, l_A, \qquad {}^4\bar{\overset{\circ}{E}}{}^{(a)}_A = {}^4\bar{E}_{(a)A} = (\bar{n}_{(a)}; {}^3\bar{e}_{(a)r}).
\end{aligned}
\tag{2.3}
$$

The future-oriented unit normal to Σ_τ and the projector on Σ_τ are

$$
\begin{aligned}
l_A &= \epsilon\,(1+n)\Big(1;\, 0\Big), \qquad {}^4g^{AB}\, l_A\, l_B = \epsilon,\\
l^A &= \epsilon\,(1+n)\,{}^4g^{A\tau} = \frac{1}{1+n}\Big(1;\, -n^r\Big) = \frac{1}{1+n}\Big(1;\, -\bar{n}_{(a)}\,{}^3\bar{e}^r_{(a)}\Big),\\
{}^3h^B_A &= \delta^B_A - \epsilon\, l_A\, l^B, \qquad {}^3h^\tau_\tau = {}^3h^\tau_r = 0, \quad {}^3h^r_\tau = \bar{n}_{(a)}\,{}^3\bar{e}^r_{(a)}, \quad {}^3h^r_s = \delta^r_s,\\
&\qquad {}^3h_{\tau\tau} = -\epsilon \sum_a \bar{n}^2_{(a)}, \quad {}^3h_{\tau r} = -\epsilon\, \bar{n}_{(a)}\,{}^3\bar{e}_{(a)r}, \quad {}^3h_{rs} = -\epsilon\,{}^3\bar{e}_{(a)r}\,{}^3\bar{e}_{(a)s},\\
&\qquad {}^3h^{\tau\tau} = {}^3h^{\tau r} = 0, \qquad {}^3h^{rs} = -\epsilon\,{}^3\bar{e}^r_{(a)}\,{}^3\bar{e}^s_{(a)}.
\end{aligned}
\tag{2.4}
$$

The 4-metric has the following expression (the lapse and shift function are independent inertial gauge variables)

$$
\begin{aligned}
{}^4g_{\tau\tau} &= \epsilon\,[(1+n)^2 - {}^3g^{rs}\, n_r\, n_s] = \epsilon\,[(1+n)^2 - \sum_a n^2_{(a)}] = \epsilon\,[(1+n)^2 - \sum_a \bar{n}^2_{(a)}],\\
{}^4g_{\tau r} &= -\epsilon\, n_r = -\epsilon\, n_{(a)}\,{}^3e_{(a)r} = -\epsilon\,\bar{n}_{(a)}\,{}^3\bar{e}_{(a)r},\\
{}^4g_{rs} &= -\epsilon\,{}^3g_{rs},\\
&\qquad {}^3g_{rs} = {}^3e_{(a)r}\,{}^3e_{(a)s} = {}^3\bar{e}_{(a)r}\,{}^3\bar{e}_{(a)s}, \qquad {}^3g^{rs} = {}^3h^{rs} = {}^3e^r_{(a)}\,{}^3e^s_{(a)} = {}^3\bar{e}^r_{(a)}\,{}^3\bar{e}^s_{(a)},\\
{}^4g^{\tau\tau} &= \frac{\epsilon}{(1+n)^2}, \qquad {}^4g^{\tau r} = -\epsilon\,\frac{n^r}{(1+n)^2} = -\epsilon\,\frac{{}^3e^r_{(a)}\, n_{(a)}}{(1+n)^2} = -\epsilon\,\frac{{}^3\bar{e}^r_{(a)}\,\bar{n}_{(a)}}{(1+n)^2},\\
{}^4g^{rs} &= -\epsilon\,({}^3g^{rs} - \frac{n^r\, n^s}{(1+n)^2}) = -\epsilon\,{}^3e^r_{(a)}\,{}^3e^s_{(b)}\,(\delta_{(a)(b)} - \frac{n_{(a)}\, n_{(b)}}{(1+n)^2}) =\\
&= -\epsilon\,{}^3\bar{e}^r_{(a)}\,{}^3\bar{e}^s_{(b)}\,(\delta_{(a)(b)} - \frac{\bar{n}_{(a)}\,\bar{n}_{(b)}}{(1+n)^2}), \qquad {}^3g^{rs} = h^{rs},\\
&\qquad {}^4g^{\tau\tau}\,{}^4g^{rs} - {}^4g^{\tau r}\,{}^4g^{\tau s} = -\frac{{}^3g^{rs}}{(1+n)^2}, \qquad {}^3g = \gamma = ({}^3e)^2, \quad {}^3e = det\,{}^3e_{(a)r},\\
&\qquad \sqrt{-g} = \sqrt{|{}^4g|} = \frac{\sqrt{{}^3g}}{\sqrt{\epsilon\,{}^4g^{\tau\tau}}} = \sqrt{\gamma}\,(1+n) = {}^3e\,(1+n).
\end{aligned}
\tag{2.5}
$$

The 3-metric ${}^3g_{rs}$ has signature $(+++)$, so that we may put all the flat 3-indices *down*. We have ${}^3g^{ru}\, {}^3g_{us} = \delta^r_s$, $\partial_A\, {}^3g^{rs} = -{}^3g^{ru}\, {}^3g^{sv}\, \partial_A\, {}^3g_{uv}$.

C. The Two Congruences of Time-like Observers associated with Non-Inertial Frames

Each 3+1 splitting of space-time, i.e. each global non-inertial frame, has two associated congruences of time-like observers:

i) The congruence of the Eulerian observers with the unit normal $l^\mu(\tau,\vec{\sigma}) = \Big(z^\mu_A\, l^A\Big)(\tau,\vec{\sigma})$ to the 3-spaces as unit 4-velocity. The world-lines of these observers are the integral curves of the unit normal and in general are not geodesics. In adapted radar 4-coordinates the contro-variant $(l^A(\tau,\vec{\sigma}),\, {}^4\overset{\circ}{\bar{E}}{}^A_{(a)}(\tau,\vec{\sigma}))$ and covariant $(l_A(\tau,\vec{\sigma}),\, {}^4\overset{\circ}{\bar{E}}_{(a)A}(\tau,\vec{\sigma}))$ orthonormal tetrads carried by the Eulerian observers are given in Eqs.(2.3).

ii) The skew congruence with unit 4-velocity $v^\mu(\tau,\vec{\sigma}) = \Big(z^\mu_A\, v^A\Big)(\tau,\vec{\sigma})$ (in general it is not surface-forming, i.e. it has a non-vanishing vorticity). The observers of the skew congruence have the world-lines (integral curves of the 4-velocity) defined by $\sigma^r = const.$ for every τ, because the unit 4-velocity tangent to the flux lines $x^\mu_{\vec{\sigma}_o}(\tau) = z^\mu(\tau,\vec{\sigma}_o)$ is $v^\mu_{\vec{\sigma}_o}(\tau) = z^\mu_\tau(\tau,\vec{\sigma}_o)/\sqrt{\epsilon\, {}^4g_{\tau\tau}(\tau,\vec{\sigma}_o)}$. They carry the adapted contro-variant orthonormal tetrads ($\mathcal{V}^A_{(a)}(\tau,\vec{\sigma})$ are not tangent to the 3-spaces Σ_τ like ${}^4\overset{\circ}{\bar{E}}{}^A_{(a)}(\tau,\vec{\sigma})$ of Eqs.(2.3))

$$
\begin{aligned}
v^A(\tau,\vec{\sigma}) &= \frac{(1;0)}{\sqrt{(1+n)^2 - \sum_a \bar{n}^2_{(a)}}}(\tau,\vec{\sigma}),\\
\mathcal{V}^A_{(a)}(\tau,\vec{\sigma}) &= \Big(\frac{\bar{n}_{(a)}}{(1+n)^2}; (\delta_{(a)(b)} - \frac{\bar{n}_{(a)}\,\bar{n}_{(b)}}{(1+n)^2})\, {}^3\bar{e}^r_{(b)}\Big)(\tau,\vec{\sigma}).
\end{aligned}
\tag{2.6}
$$

The covariant version of these tetrads is

$$
\begin{aligned}
\epsilon\, v_A(\tau,\vec{\sigma}) &= \Big(\sqrt{(1+n)^2 - \sum_c \bar{n}^2_{(c)}}; \frac{-\bar{n}_{(a)}\, {}^3\bar{e}_{(a)r}}{\sqrt{(1+n)^2 - \sum_c \bar{n}^2_{(c)}}}\Big)(\tau,\vec{\sigma}),\\
\mathcal{V}_{(a)A}(\tau,\vec{\sigma}) &= \Big(0; {}^3\bar{e}_{(a)r}\Big)(\tau,\vec{\sigma}).
\end{aligned}
\tag{2.7}
$$

In each point there is a Lorentz transformation connecting these tetrads to the ones of the Eulerian observer present in this point.

When there is a perfect fluid with unit time-like 4-velocity $U^A(\tau,\vec{\sigma})$, there is also the congruence of the time-like flux curves: in general it is not surface-forming and it is independent from the previous two congruences. If $\Big(U^A(\tau,\vec{\sigma}); \mathcal{U}^A_{(a)}(\tau,\vec{\sigma})\Big)$ is an orthonormal tetrad carried by a flux line, the connection of these 4-vectors to the orthonormal tetrad of the Eulerian observers [5] is

$$\begin{aligned} U^A(\tau,\vec{\sigma}) &= \Gamma\Big(l^A + \sum_a \beta_{(a)}\, {}^4\bar{\overset{\circ}{E}}{}^A_{(a)}\Big)(\tau,\vec{\sigma}),\\ \mathcal{U}^A_{(a)}(\tau,\vec{\sigma}) &= \Big(t_{(a)}\, l^A + \sum_b \gamma_{(a)(b)}\, {}^4\bar{\overset{\circ}{E}}{}^A_{(b)}\Big)(\tau,\vec{\sigma}), \end{aligned} \tag{2.8}$$

with $t_{(a)}(\tau,\vec{\sigma}) = \Big(\sum_b \gamma_{(a)(b)}\, \beta_{(b)}\Big)(\tau,\vec{\sigma})$ and $\Big[\sum_{cd}\Big(\delta_{(c)(d)} - \beta_{(c)}\,\beta_{(d)}\Big)\,\gamma_{(a)(c)}\,\gamma_{(b)(d)}\Big](\tau,\vec{\sigma}) = \delta_{(a)(b)}$. See Section V for the gauge fixings implying $U^A(\tau,\vec{\sigma}) \approx l^A(\tau,\vec{\sigma})$.

In Ref.[8] (see also Refs. [14]) the results of Ref.[7] are reformulated in arbitrary coordinates with the following change of notation

$$\begin{aligned} U^A &= \Gamma(w)\Big(l^A + m^A\Big) = \Gamma(w)\Big(1; -[\sum_a \bar{n}_{(a)}\, {}^3\bar{e}^r_{(a)} + (1+n)\, w^r]\Big),\\ &\quad m^A = \sum_a \beta_{(a)}\, {}^4\bar{\overset{\circ}{E}}{}^A_{(a)} = \Big(0; -w^r = \sum_a \beta_{(a)}\, {}^3\bar{e}^r_{(a)}\Big),\\ U_A &= \Gamma(w)\Big(l_A + m_A\Big) = \epsilon\,\Gamma(w)\Big(1 + n + \sum_{as} \bar{n}_{(a)}\, {}^3\bar{e}^s_{(a)}\, w_s; w_r\Big),\\ &\quad m_A = \epsilon\Big(\sum_{as} \bar{n}_{(a)}\, {}^3\bar{e}^s_{(a)}\, w_s; w_r\Big), \qquad m_A\, l^A = 0,\\ &\quad \Gamma(w) = \frac{1}{\sqrt{1 - {}^3g^{rs} w_r\, w_s}} = \frac{1}{\sqrt{1 - {}^3g_{rs} w^r\, w^s}} = \sqrt{1 + {}^3g^{rs}\, U_r\, U_s},\\ &\quad w_r = \epsilon\, \frac{U_r}{\sqrt{1 + {}^3g^{rs}\, U_r\, U_s}}, \qquad w^r = {}^3g^{rs}\, w_s. \end{aligned} \tag{2.9}$$

When the vorticity of the fluid vanishes, so that its 4-velocity is surface forming, there is a 3+1 splitting of space-time determined by the irrotational fluid. While in special relativity we can always choose a global non-inertial frame coinciding with these 3+1 splitting, in general relativity we have to show that there is a gauge fixing on the inertial gauge variable ${}^3K(\tau,\vec{\sigma})$ (the York time) allowing this identification (see Section V).

Let us remark that Eqs.(2.8) establish the bridge between our 3+1 point of view and the 1+3 point of view of Refs.[15], where one describes both the gravitational field and the matter as seen by a generic family of observers with 4-velocity $U^A(\tau,\vec{\sigma})$. Most of the results in cosmology (see for instance Refs.[16]) are presented in the 1+3 framework. However, in the 1+3 point of view vorticity is an obstruction to formulate the Cauchy problem (3-spaces

[5] Analogous expressions hold in terms of the tetrads (2.6) of the observers of the skew congruence. See Eqs.(5.3) for the gauge fixings implying $U^A(\tau,\vec{\sigma}) \approx v^A(\tau,\vec{\sigma})$.

are not existing; each observer uses as rest frame the tangent 3-space orthogonal to the 4-velocity) and there is no natural way to identify the inertial gravitational gauge variables of the Hamiltonian formalism based on Dirac's constraint theory (see also Appendix A of paper I for the induced treatment of the non-Hamiltonian ADM equations).

III. PERFECT FLUIDS IN MINKOWSKI SPACE-TIME

In this Section we will review Brown approach [9] to perfect fluids in Minkowski space-time. In Ref.[9] a set of space-time scalar fields $\tilde{\alpha}^i(x)$, $i = 1, 2, 3$, was interpreted as *Lagrangian (or comoving) coordinates for the fluid*. They label the fluid flow lines (physically determined by the average particle motions) passing through the points inside the boundary. In an allowed 3+1 splitting of Minkowski space-time, we use the three scalar fields $\alpha^i(\tau, \vec{\sigma}) = \tilde{\alpha}^i(z(\tau, \vec{\sigma}))$ for the Lagrangian coordinates of the fluid in the instantaneous 3-space Σ_τ. Now either the $\alpha^i(\tau, \vec{\sigma})$'s have a compact boundary $V_\alpha(\tau) \subset \Sigma_\tau$ or have boundary conditions at spatial infinity. For each value of τ, one could invert $\alpha^i = \alpha^i(\tau, \vec{\sigma})$ to $\sigma^r = \sigma^r(\tau, \alpha^i)$ and use the α^i's as a special coordinate system on Σ_τ inside the support $V_\alpha(\tau) \subset \Sigma_\tau$: $z^\mu(\tau, \vec{\sigma}(\tau, \alpha^i)) = \check{z}^\mu(\tau, \alpha^i)$.

The *unit time-like 4-velocity field* $U^A(\tau, \vec{\sigma})$ of the fluid is a derived function of the Lagrangian coordinates of the fluid. In this way, as shown in Ref.[9], one has not to add new quantities (like the Clebsch potentials) with Lagrange multipliers to the action to implement the particle number conservation and the absence of entropy exchange between neighboring flow lines: they are automatically satisfied as a consequence of the comoving nature of these Lagrangian coordinates, which implies

$$\Big(U^A\, \partial_A\, \alpha^i\Big)(\tau, \vec{\sigma}) = 0. \tag{3.1}$$

Instead of the 4-velocity it is convenient to introduce a *material current* [9, 10] $J^A(\alpha^i(\tau, \vec{\sigma})) = \sqrt{|{}^4g(\tau, \vec{\sigma})|}\, \tilde{n}(\tau, \vec{\sigma})\, U^A(\tau, \vec{\sigma})$, $\Big(J^A\, \partial_A\, \alpha^i\Big)(\tau, \vec{\sigma}) = 0$, where $\tilde{n}(\tau, \vec{\sigma})$ is the particle number density. This material current has the following form [6]

$$\begin{aligned}
J^A(\alpha^i(\tau, \vec{\sigma})) &= [(1+n)\, \sqrt{\gamma}\, \tilde{n}\, U^A](\tau, \vec{\sigma}), \qquad [J^\tau\, \partial_\tau\, \alpha^i + J^r\, \partial_r\, \alpha^i](\tau, \vec{\sigma}) = 0,\\
J^\tau(\alpha^i(\tau, \vec{\sigma})) &= [\epsilon^{ruv}\, \partial_r\, \alpha^1\, \partial_u\, \alpha^2\, \partial_v\, \alpha^3](\tau, \vec{\sigma}) =\\
&= \frac{1}{6}\, \epsilon^{ruv}\, \epsilon_{ijk}\, [\partial_r\, \alpha^i\, \partial_u\, \alpha^j\, \partial_v\, \alpha^k](\tau, \vec{\sigma}) = det\, (\partial_r\, \alpha^i)(\tau, \vec{\sigma}),\\
J^r(\alpha^i(\tau, \vec{\sigma})) &= -\frac{1}{2}\, \epsilon^{ruv}\, \epsilon_{ijk}\, [\partial_\tau\, \alpha^i\, \partial_u\, \alpha^j\, \partial_v\, \alpha^k](\tau, \vec{\sigma}),\\
&\overset{(3.1)}{\Rightarrow} \qquad \partial_\tau\, \alpha^i = -\frac{J^r}{J^\tau}\, \partial_r\, \alpha^i = -\frac{U^r}{U^\tau}\, \partial_r\, \alpha^i,
\end{aligned} \tag{3.2}$$

with $\mathcal{N} = \int_{V_\alpha(\tau)} d^3\sigma\, J^\tau(\alpha^i(\tau, \vec{\sigma}))$ giving the conserved particle number and with $\int_{V_\alpha(\tau)} d^3\sigma (s\, J^\tau)(\tau, \vec{\sigma})$ giving the conserved entropy per particle.

This shows that the fluid flow lines, whose tangent vector field is the fluid 4-velocity time-like vector field U^A, are identified by $\alpha^i = const.$ and that the particle number conservation is automatic. Moreover, if the entropy for particle is a function only of the fluid Lagrangian

[6] Here J^τ has the opposite sign with respect to Ref.[10]. We have $\epsilon^{ruv}\, \epsilon_{ijk}\, \partial_m\, \alpha^i\, \partial_u\, \alpha^j\, \partial_v\, \alpha^k = 2\, \delta^r_m\, det\, (\partial_u\, \alpha^k) = 2\, J^\tau\, \delta^r_m$ and $\epsilon^{ruv}\, \epsilon_{ijk}\, \partial_r\, \alpha^h\, \partial_u\, \alpha^j\, \partial_v\, \alpha^k = 2\, J^\tau\, \delta^h_i$.

coordinates, $s = s(\alpha^i)$, the assumed form of J^A also implies automatically the absence of entropy exchange between neighboring flow lines, $\partial_A(\tilde{n}\, s\, U^A) = 0$. Since $U^A\,\partial_A s(\alpha^i) = 0$, the perfect fluid is locally adiabatic; instead for an isentropic fluid we have $\partial_A s = 0$, namely $s = const.$.

Since U^A is a unit 4-vector, we have the following expression for the particle number density and for the unit 4-velocity in terms of the material current

$$
\begin{aligned}
\tilde{n}(\tau,\vec{\sigma}) &= \frac{|J|}{(1+n)\,\sqrt{\gamma}}(\tau,\vec{\sigma}) = \frac{\sqrt{\epsilon\, {}^4g_{AB}(\tau,\vec{\sigma})\, J^A(\alpha^i(\tau,\vec{\sigma}))\, J^B(\alpha^i(\tau,\vec{\sigma}))}}{|{}^4g(\tau,\vec{\sigma})|} = \\
&= \frac{1}{\sqrt{\gamma(\tau,\vec{\sigma})}}\sqrt{(J^\tau)^2 - {}^3g_{rs}\,\frac{J^r + n^r\, J^\tau}{1+n}\,\frac{J^s + n^s\, J^\tau}{1+n}}(\tau,\vec{\sigma}),\\
U^A(\tau,\vec{\sigma}) &= \frac{J^A(\alpha^i(\tau,\vec{\sigma}))}{\sqrt{\epsilon\, {}^4g_{EF}(\tau,\vec{\sigma})\, J^E(\alpha^i(\tau,\vec{\sigma}))\, J^F(\alpha^i(\tau,\vec{\sigma}))}}.
\end{aligned}
\tag{3.3}
$$

This unit 4-velocity is in general different from the unit normal $l^A(\tau,\vec{\sigma})$ to the 3-space Σ_τ and therefore in general it is not surface forming.

The action given in Ref.[9] for an (isentropic if $s = const.$) perfect fluid with equation of state $\rho = \rho(\tilde{n}, s)$ [7], can be reformulated as a parametrized Minkowski theory [10, 12] if written in the form

$$
\begin{aligned}
S_{fluid} &= \int d\tau d^3\sigma\, L({}^4g_{AB}(\tau,\vec{\sigma}), \alpha^i(\tau,\vec{\sigma})) = \\
&= -\int d\tau d^3\sigma\, ((1+n)\,\sqrt{\gamma})(\tau,\vec{\sigma})\, \rho\Big(\frac{|J(\alpha^i(\tau,\vec{\sigma}))|}{((1+n)\,\sqrt{\gamma})(\tau,\vec{\sigma})},\, s(\alpha^i(\tau,\vec{\sigma}))\Big) = \\
&= -\int d\tau d^3\sigma\, ((1+n)\,\sqrt{\gamma})(\tau,\vec{\sigma})\\
&\quad \rho\Big(\frac{1}{\sqrt{\gamma(\tau,\vec{\sigma})}}\sqrt{\Big[(J^\tau)^2 - {}^3g_{uv}\,\frac{J^u + n^u\, J^\tau}{1+n}\,\frac{J^v + n^v\, J^\tau}{1+n}\Big]}(\tau,\vec{\sigma}),\, s(\alpha^i(\tau,\vec{\sigma}))\Big).
\end{aligned}
\tag{3.4}
$$

Here S_{fluid} is a functional of the fluid and of the embedding $z^\mu(\tau,\vec{\sigma})$. See Ref.[12] for the general treatment of the parametrized Minkowski theories, for the gauge equivalence of the descriptions in different admissible 3+1 splittings of Minkowski space-time and for rest-frame instant form description. Here we only emphasize the properties of the fluid developed in Ref.[10] in view of the extension to general relativity.

The pressure of the fluid is

$$
p(\tilde{n}, s) = \tilde{n}\,\frac{\partial\,\rho(\tilde{n}, s)}{\partial\tilde{n}}|_s - \rho(\tilde{n}, s).
\tag{3.5}
$$

[7] As shown in Ref.[9], in the case that the entropy has the form $s = s(\alpha^i)$ the resulting Euler-Lagrange equations imply the standard Euler equations obtained from the conservation of the energy-momentum tensor. See also Ref.[10].

The canonical momenta of the fluid are

$$\Pi_i(\tau,\vec{\sigma}) \;=\; -\frac{\delta\, S_{fluid}}{\delta\,\partial_\tau\,\alpha^i(\tau,\vec{\sigma})} = \frac{Y^r(\tau,\vec{\sigma})}{X(\tau,\vec{\sigma})}\, T_{ri}(\tau,\vec{\sigma})\, \Big(\frac{\partial\,\rho(\tilde{n},s)}{\partial\,\tilde{n}}\Big)|_{\tilde{n}=\frac{X}{\sqrt{\gamma}}}(\tau,\vec{\sigma}),$$

with the notations

$$X = |J| = \sqrt{(J^\tau)^2 - {}^3g_{uv}\,Y^u\,Y^v} = \sqrt{\gamma}\,\tilde{n},\qquad Y^u = \frac{J^u + n^u\,J^\tau}{1+n},$$

$$\begin{aligned}
\mathcal{T}_{ri} \;&=\; \frac{1}{2}\,{}^3g_{rs}\,\epsilon^{suv}\,\epsilon_{ijk}\,\partial_u\,\alpha^j\,\partial_v\,\alpha^k,\qquad \mathcal{T}_{ri}\,\partial_n\,\alpha^i = {}^3g_{rn}\,J^\tau,\\
(\mathcal{T}^{-1})^{is} \;&=\; \frac{{}^3g^{sn}\,\partial_n\,\alpha^i}{J^\tau},\qquad \partial_n\,\alpha^i\,\Pi_i = J^\tau\,{}^3g_{nr}\,\frac{Y^r}{X}\,\Big(\frac{\partial\,\rho(\tilde{n},s)}{\partial\,\tilde{n}}\Big)|_{\tilde{n}=\frac{X}{\sqrt{\gamma}}},\\
\Rightarrow\qquad & Y^r = \Pi_i\,(\mathcal{T}^{-1})^{ir}\,\frac{X}{\Big(\frac{\partial\,\rho(\tilde{n},s)}{\partial\,\tilde{n}}\Big)|_{\tilde{n}=\frac{X}{\sqrt{\gamma}}}} = \frac{1}{J^\tau}\,{}^3g^{rs}\,\partial_s\,\alpha^i\,\Pi_i\,\frac{X}{\Big(\frac{\partial\,\rho(\tilde{n},s)}{\partial\,\tilde{n}}\Big)|_{\tilde{n}=\frac{X}{\sqrt{\gamma}}}} = Y^r(X,\Pi_i).
\end{aligned}\tag{3.6}$$

The matrix $\mathcal{T}_{ri}$ is equal to $\sum_s\,{}^3g_{rs}\,(ad\,\mathcal{J}_{is})$, where $ad\,\mathcal{J}_{ir} = J^\tau\,\mathcal{J}^{-1}_{ir}$ is the adjoint matrix of the Jacobian $\mathcal{J} = (\mathcal{J}_{ir} = \partial_r\,\alpha^i)$ of the transformation from the Lagrangian coordinates $\alpha^i(\tau,\vec{\sigma})$ to the Eulerian ones $\vec{\sigma}$ on Σ_τ (see Ref.[11] and Section III). We use the notations $J^\tau = (det\,\mathcal{J})$, $\sum_i\,\mathcal{I}_{ir}\,\mathcal{I}^{-1}_{is} = \delta_{rs}$, $\sum_r\,\mathcal{I}_{ir}\,\mathcal{I}^{-1}_{jr} = \delta_{ij}$, $\sum_r\,(\mathcal{T}^{-1})^{ir}\,\mathcal{T}_{rj} = \delta^i_j$, $\sum_i\,\mathcal{T}_{ri}\,(\mathcal{T}^{-1})^{is} = \delta^s_r$.

The main problem of this approach is to invert these equations to get the velocities $\partial_\tau\,\alpha^i(\tau,\vec{\sigma})$ in terms of the momenta $\Pi_i(\tau,\vec{\sigma})$, i.e. to get the unit 4-velocity $U^A(\tau,\vec{\sigma})$ and then the energy-momentum tensor and the Hamiltonian in terms of the momenta.

We have to find the solution X of the equation $X^2 + {}^3g_{rs}\,Y^r(X,\Pi_i)\,Y^s(X,\Pi_i) = (J^\tau)^2$ with $Y^r(X,\Pi_i)$ given by the last line of Eq.(3.6). This equation can be rewritten in the following form

$$X^2\,\Big[(J^\tau)^2 + A^2\,\Big(\frac{\partial\,\rho(\tilde{n},s)}{\partial\,\tilde{n}}\Big)^{-2}|_{\tilde{n}=\frac{X}{\sqrt{\gamma}}}\Big] = (J^\tau)^4,\qquad A^2 = \sum_{rs}\,{}^3g^{rs}\,\partial_r\,\alpha^i\,\Pi_i\,\partial_s\,\alpha^j\,\Pi_j.\tag{3.7}$$

For every equation of state $\rho = \rho(\tilde{n},s)$ the solution of this equation has the form $X = \tilde{X}(\Pi_i) = \sqrt{\gamma}\,\tilde{n} = F(\sqrt{\gamma}, A^2, (J^\tau)^2)[\rho]$, with all the dependence upon the fluid momenta in the quadratic form A^2 and with an extra dependence on $J^\tau = det\,(\partial_r\,\alpha^i)$. See Appendix B for the known solutions of Eq.(3.7).

Once $X = \tilde{X}(\Pi_i)$ is known, we have

$$\begin{aligned}
Y^r \;&=\; Y^r(X,\Pi_i) = \tilde{Y}^r(\Pi_i),\qquad J^r = (1+n)\,\tilde{Y}^r(\Pi_i) - n^r\,J^\tau,\qquad \partial_\tau\,\alpha^i = -\frac{\tilde{J}^r(\Pi_i)}{J^\tau}\,\partial_r\,\alpha^i,\\
U^\tau \;&=\; \tilde{U}^\tau(\Pi_i) = \frac{J^\tau}{(1+n)\,\tilde{X}(\Pi_i)},\qquad U^r = \tilde{U}^r(\Pi_i) = \frac{\tilde{J}^r(\Pi_i)}{(1+n)\,\tilde{X}(\Pi_i)}.
\end{aligned}\tag{3.8}$$

The energy-momentum tensor of the fluid (see Eq.(4.2) of Ref.[10]) is

$$
\begin{aligned}
T^{AB}(\tau,\vec{\sigma})[\alpha] &= -\Big[\frac{2}{\sqrt{g}}\,\frac{\delta S}{\delta\,{}^4g_{AB}}\Big](\tau,\vec{\sigma}) = \\
&= \Big[\rho\,{}^4g^{AB} - \tilde{n}\,\frac{\partial\rho}{\partial\,\tilde{n}}|_s\,({}^4g^{AB} - \frac{J^A\,J^B}{{}^4g_{CD}\,J^C\,J^D})\Big](\tau,\vec{\sigma}) = \\
&= \Big[\epsilon\,\rho\,U^A\,U^B - p\,({}^4g^{AB} - \epsilon\,U^A\,U^B)\Big](\tau,\vec{\sigma}),
\end{aligned}
\tag{3.9}
$$

As shown in Eq.(3.11) of Ref.[2] it is convenient to express its components $T^{\tau\tau}$ and $T^{\tau r}$ in terms of a mass density $\mathcal{M}$ and a mass current density $\mathcal{M}_r$ in the following way

$$
\begin{aligned}
T^{\tau\tau}(\tau,\vec{\sigma}) &= \epsilon\,(\rho+p)\,(U^\tau)^2 - p\,{}^4g^{\tau\tau} = \Big(\frac{\mathcal{M}}{(1+n)^2\,\sqrt{\gamma}}\Big)(\tau,\vec{\sigma}),\\
T^{\tau r}(\tau,\vec{\sigma}) &= \epsilon\,(\rho+p)\,U^\tau\,U^r - p\,{}^4g^{\tau r} = \Big(\frac{(1+n)\,{}^3g^{rs}\,\mathcal{M}_s - n^r\,\mathcal{M}}{(1+n)^2\,\sqrt{\gamma}}\Big)(\tau,\vec{\sigma}),
\end{aligned}
\tag{3.10}
$$

As shown in Ref.[10], by using Eqs.(3.5) and (3.6), the mass density and the mass current density of the fluid have the expression

$$
\begin{aligned}
\mathcal{M}(\tau,\sigma) &= \Big(\sqrt{\gamma}\,(1+n)^2\,T^{\tau\tau}\Big)(\tau,\vec{\sigma}) = \\
&= \Big(\frac{{}^3g_{rs}\,Y^r(X,\Pi_i)\,Y^s(X,\Pi_i)}{X}\,\frac{\partial\,\rho(\tilde{n},s)}{\partial\,\tilde{n}}|_{\tilde{n}=\frac{X}{\sqrt{\gamma}}} + \sqrt{\gamma}\,\rho(\frac{X}{\sqrt{\gamma}},s)\Big)(\tau,\vec{\sigma}) = \\
&= \Big(\frac{{}^3g^{rs}\,\partial_r\,\alpha^i\,\partial_s\,\alpha^j\,\Pi_i\,\Pi_j\;X}{(J^\tau)^2\,\frac{\partial\,\rho(\tilde{n},s)}{\partial\,\tilde{n}}|_{\tilde{n}=\frac{X}{\sqrt{\gamma}}}} + \sqrt{\gamma}\,\rho(\frac{X}{\sqrt{\gamma}},s)\Big)(\tau,\vec{\sigma}),\\
\mathcal{M}_r(\tau,\sigma) &= \Big(\sqrt{\gamma}\,(1+n)\,{}^3g_{rs}\,\Big[T^{\tau s} + n^s\,T^{\tau\tau}\Big]\Big)(\tau,\vec{\sigma}) = \Big(\partial_r\,\alpha^i\,\Pi_i\Big)(\tau,\vec{\sigma}).
\end{aligned}
\tag{3.11}
$$

The mass current density has a universal Hamiltonian expression independent from the equation of state. Instead one can get the mass density $\mathcal{M}(\tau,\vec{\sigma})$ explicitly in terms of the momenta only for those equations of state (like the dust and the photon gas) allowing to find an explicit solution of Eq.(3.7). Therefore in the rest of the paper we shall consider only the coupling of the dust to ADM tetrad gravity.

Let us remark that Eqs.(2.8)-(2.9) imply that the energy-momentum tensor (3.9) of a perfect fluid looks like the one of a viscous fluid to the Eulerian observers ($U^A = \Gamma(w)\,(l^A + m^A)$)

$$
T^{AB} = \epsilon\,\tilde{\rho}\,l^A\,l^B - \tilde{p}\,({}^4g^{AB} - \epsilon\,l^A\,l^B) + l^A\,q^B + l^B\,q^A + \pi^{AB},
\tag{3.12}
$$

where $\tilde{\rho} = \Gamma^2(w)\,\rho$, $\tilde{p} = \Gamma^2(w)\,p$, $q^A = \epsilon\,\Gamma^2(w)\,(\rho+p)\,m^A$ and $\pi^{AB} = \epsilon\,\Gamma^2(w)\,(\rho+p)\,m^A\,m^B + (\Gamma^2(w)-1)\,p^4g^{AB}$. For a viscous fluid $\tilde{\rho}$ is the matter energy density, $\tilde{p}$ the effective isotropic pressure (equilibrium pressure plus the associated bulk viscosity), q^A the total energy flux vector and π^{AB} the symmetric trace-free anisotropic stress tensor.

A. The Perfect Fluid Coupled to Gravity and the Bianchi Identities for the Energy-Momentum Tensor

When the perfect fluid is coupled to gravity, we have to reinterpret the metric appearing in the action (3.14) as the 4-metric describing the gravitational field. If we add the ADM action to Eq.(3.14) (see the next Section for the case of dust)), then we get the Einstein equations ${}^4G^{AB} = \frac{8\pi G}{c^4} T^{AB}$ with the energy-momentum tensor of Eqs.(3.10) and (3.11). Then the Bianchi identities imply ${}^4\nabla_A T^{AB} \equiv 0$. As shown in Ref.[11], these identities may be written in the form

$$
\begin{aligned}
&U^B \Big({}^4\nabla_A T^A{}_B \Big) = 0,\\
&\Big(\delta_A{}^B - U_A U^B \Big) {}^4\nabla_C T^C{}_B = 0.
\end{aligned} \tag{3.13}
$$

The first line of Eqs.(3.13) implies the equation

$$
U^A \frac{\partial \rho}{\partial \sigma^A} + (\rho + p)\, {}^4\nabla_A U^A = 0, \tag{3.14}
$$

whose final form

$$
\frac{\partial \rho}{\partial \tilde n} {}^4\nabla_A(\, \tilde n\, U^A\,) + \frac{\partial \rho}{\partial s} U^A \frac{\partial s}{\partial \sigma^A} = 0, \tag{3.15}
$$

is obtained by using Eq.(3.5). This equation is automatically satisfied when the particles number conservation law ${}^4\nabla_A(\, \tilde n\, U^A\,) = 0$ and the entropy conservation law $U^A \frac{\partial s}{\partial \sigma^A} = 0$ are satisfied.

Only three of the four equations in the second line of Eqs.(3.13) are independent: they can be rewritten in the form

$$
U^B\, {}^4\nabla_B U_A = \frac{1}{\rho + p} \Big(\delta_A{}^B - U_A U^B \Big) \frac{\partial p}{\partial \sigma^B}. \tag{3.16}
$$

and turn out to be the *relativistic Euler equations*. They allow to express the *acceleration* $a_{(U)A} = U^B\, {}^4\nabla_B U_A$ as a function of "internal forces" depending on the pressure (for the dust we will get $a_{(U)A} = 0$ in Eq.(4.19))

$$
a_{(U)A} = \frac{1}{\rho + p} \Big(\delta_A{}^B - U_A U^B \Big) \frac{\partial p}{\partial \sigma^B}. \tag{3.17}
$$

Finally, by using the notations of Appendix A for tetrad gravity and Eqs.(3.11) of Ref.[2], the identities ${}^4\nabla_A T^{AB} \equiv 0$ may be written in the following form

$$\Big[\partial_\tau - \sum_{ar} {}^3e^r_{(a)}\, n_{(a)}\, \partial_r - (1+n)\, {}^3K\Big]\, (({}^3e)^{-1}\, \mathcal{M}) + \frac{1+n}{{}^3e} \sum_s \partial_s \Big[\sum_{ar} {}^3e^s_{(a)}\, {}^3e^r_{(a)}\, \mathcal{M}_r \Big] +$$
$$+2\, ({}^3e)^{-1} \sum_{ars} \partial_s n\, {}^3e^s_{(a)}\, {}^3e^r_{(a)}\, \mathcal{M}_r - (1+n) \sum_{abrsuv} {}^3K_{rs}\, {}^3e^r_{(a)}\, {}^3e^s_{(b)}\, {}^3e^u_{(a)}\, {}^3e^v_{(b)}\, T_{uv} \equiv 0,$$

$$\begin{aligned}
&\Big[\partial_\tau - \sum_{as} {}^3e^s_{(a)}\, n_{(a)}\, \partial_s\Big]\, (({}^3e)^{-1}\, \mathcal{M}_r) + \partial_r\, n\, ({}^3e)^{-1}\, \mathcal{M} - \\
&- \sum_{as} \partial_r\, ({}^3e^s_{(a)}\, n_{(a)})\, ({}^3e)^{-1}\, \mathcal{M}_s - (1+n)\, {}^3K\, ({}^3e)^{-1}\, \mathcal{M}_r - \\
&- \sum_{auv} \partial_u\, n\, {}^3e^u_{(a)}\, {}^3e^v_{(a)}\, T_{rv} + (1+n)\, \Big[\frac{1}{{}^3e} \sum_{auv} \partial_u \Big({}^3e\, {}^3e^u_{(a)}\, {}^3e^s_{(a)}\, T_{sr}\Big) - \\
&- \frac{1}{2} \sum_{abcuvmn} \partial_r\, ({}^3e_{(c)u}\, {}^3e_{(c)v})\, {}^3e^u_{(a)}\, {}^3e^v_{(b)}\, {}^3e^m_{(a)}\, {}^3e^n_{(b)}\, T_{mn}\Big] \equiv 0.
\end{aligned} \qquad (3.18)$$

Since Eqs.(2.9) imply the following form of Eqs.(3.11)

$$\begin{aligned}
({}^3e)^{-1}\, \mathcal{M} &= \Gamma^2(w)\, (\rho + p) - p, \\
({}^3e)^{-1}\, \mathcal{M}_r &= -\Gamma^2(w)\, (\rho + p)\, w_r,
\end{aligned} \qquad (3.19)$$

the use of Eqs.(3.9) allows to get the following expression of Eqs.(3.18) as equations for w_r and ρ

$$\begin{aligned}
&\Big(\frac{\partial\, w_r}{\partial\, \tau} - \sum_s n^s\, \frac{\partial\, w_r}{\partial\, \sigma^s} - \sum_s w_s\, \frac{\partial\, n_s}{\partial\, \sigma^r}\Big) - (1+n) \sum_u w^u\, {}^3\nabla_u\, w_r - \frac{\partial\, n}{\partial\, \sigma^r} + \\
&+\ w_r\, \Big(\sum_u w^u\, \frac{\partial\, n}{\partial\, \sigma^u} + (1+n) \sum_{uv} w^u\, w^v\, {}^3K_{uv}\Big) + \\
&+\ \frac{1}{\Gamma^2(w)\, (\rho+p)}\, \Big[-(1+n)\, \frac{\partial\, p}{\partial\, \sigma^r} + w_r\, \Big(\frac{\partial\, p}{\partial\, \tau} - \sum_u n^u\, \frac{\partial\, p}{\partial\, \sigma^u}\Big)\Big] \equiv 0,
\end{aligned}$$

$$\begin{aligned}
&\Big[\frac{\partial}{\partial\, \tau} - \sum_s \Big((1+n)\, w^s + n^s\Big)\, \frac{\partial}{\partial\, \sigma^s}\Big]\, \rho - \\
&-\ (\sum_{uv} {}^3g_{uv}\, w^u\, w^v)\, \Big[\frac{\partial}{\partial\, \tau} - \sum_s n^s\, \frac{\partial}{\partial\, \sigma^s}\Big]\, p + (1+n) \sum_s w^s\, \frac{\partial\, p}{\partial\, \sigma^s} + \\
&-\ (\rho + p)\, (1+n)\, \Big[\Big({}^3K - \sum_{rs} w^r\, w^s\, {}^3K_{rs}\Big) + \sum_r {}^3\nabla_r\, w^r\Big] \equiv 0,
\end{aligned} \qquad (3.20)$$

where ${}^3\nabla$ is the covariant derivative inside the 3-spaces with 3-metric ${}^3g_{rs} = \sum_a e_{(a)r} e_{(a)s}$ and $n^r = \sum_s {}^3g^{rs}\, n_s = \sum_a n_{(a)}\, {}^3e^r_{(a)}$ are the shift functions.

IV. THE DUST IN ADM TETRAD GRAVITY

In this Section we shall consider the dust coupled to ADM tetrad gravity [1] in the globally hyperbolic, asymptotically Minkowskian space-times with boundary conditions killing the supertranslations and reducing the asymptotic symmetries to the ADM Poincare' group. The notations for the gravitational field are defined in Appendix A.

A. The Action of the Dust coupled to ADM Tetrad Gravity

Let us consider an isentropic perfect fluid, a dust with $p = 0$, $s = const.$, and equation of state $\rho = \mu\,\tilde{n} = \frac{\mu}{(1+n)\sqrt{\gamma}}\sqrt{\epsilon\,{}^4g_{AB}\,J^A\,J^B}$. In this case the chemical potential μ is the rest mass-energy of a fluid particle: $\mu = m\,c$. The full action of ADM tetrad gravity coupled to a dust (with positive energy) is

$$\begin{aligned} S &= S_{ADM} + S_{dust} = \qquad\qquad \mu = m\,c,\\ &= S_{ADM} - \int d\tau d^3\sigma\, m\,c\,\sqrt{\epsilon\,{}^4g_{AB}\,J^A\,J^B}(\tau,\vec{\sigma}), \end{aligned} \tag{4.1}$$

with S_{ADM} given in Ref.[1]. The gravitational action depends upon 16 tetradic gravitational configuration variables, which are defined in Appendix A. At the Hamiltonian level there are 16 momenta as shown in the canonical basis of Eq.(A1).

The dust momentum conjugate to α^i is ($\mathcal{T}_{ri}$ is defined in Eq.(3.6) with $\partial\,\rho/\partial\,\tilde{n} = \mu$)

$$\Pi_i(\tau,\vec{\sigma}) = -\frac{\delta\,S_{ADM}}{\delta\,\partial_\tau\alpha^i(\tau,\vec{\sigma})} = \mu\,\frac{Y^r(\tau,\vec{\sigma})}{X(\tau,\vec{\sigma})}\,\mathcal{T}_{ri}(\tau,\vec{\sigma}). \tag{4.2}$$

,

The following Poisson brackets are assumed

$$\{\alpha^i(\tau,\vec{\sigma}),\Pi_j(\tau,\vec{\sigma}^{'})\} = -\delta^i_j\,\delta^3(\vec{\sigma},\vec{\sigma}^{'}). \tag{4.3}$$

As shown in Ref. [1] there are the following ten primary first class constraints involving only the gravitational field.

$$\begin{aligned} &\pi_{\varphi_{(a)}}(\tau,\vec{\sigma}) \approx 0, \qquad \pi_n(\tau,\vec{\sigma}) \approx 0, \qquad \pi_{n_{(a)}}(\tau,\vec{\sigma}) \approx 0,\\ &{}^3M_{(a)}(\tau,\vec{\sigma}) = \epsilon_{(a)(b)(c)}\,{}^3e_{(b)r}(\tau,\vec{\sigma})\,{}^3\pi^r_{(c)}(\tau,\vec{\sigma}) \approx 0. \end{aligned} \tag{4.4}$$

Moreover there are the following four secondary first class constraints

$$
\begin{aligned}
\mathcal{H}(\tau,\vec{\sigma}) &= \mathcal{H}_{grav}(\tau,\vec{\sigma}) + \mathcal{M}(\tau,\vec{\sigma}) \approx 0,\\
\mathcal{H}_{(a)}(\tau,\vec{\sigma}) &= \mathcal{H}_{grav\,(a)}(\tau,\vec{\sigma}) + {}^3e^r_{(a)}(\tau,\vec{\sigma})\,\mathcal{M}_r(\tau,\vec{\sigma}) \approx 0.
\end{aligned}
\tag{4.5}
$$

They are the super-Hamiltonian and super-momentum constraints, which depend also on the mass density $\mathcal{M}$ and on the mass current density $\mathcal{M}_r = \partial_r\,\alpha^i\,\Pi_i$ defined in Eq.(3.11) (see Ref.[1] for the gravitational parts $\mathcal{H}_{grav}(\tau,\vec{\sigma})$, $\mathcal{H}_{grav\,(a)}(\tau,\vec{\sigma})$). Eqs.(4.5) show the necessity of having the Hamiltonian expression of the mass density $\mathcal{M}$ by solving Eq.(3.7).

For the dust we get the following solution of Eq.(3.7) (see Appendix B)

$$
\begin{aligned}
X &= \sqrt{\gamma}\,\tilde{n} = \sqrt{\gamma}\,\frac{\rho}{\mu} = \frac{\mu\,(J^\tau)^2}{\sqrt{\mu^2\,(J^\tau)^2 + A^2}} = \frac{\mu\,det^2\,(\partial_r\,\alpha^i)}{\sqrt{\mu^2\,det^2\,(\partial_r\,\alpha^i) + \sum_{uvija}\,{}^3e^u_{(a)}\,{}^3e^v_{(a)}\,\partial_u\,\alpha^i\,\partial_v\,\alpha^j\,\Pi_i\,\Pi_j}},\\
Y^r &= det\,(\partial_r\,\alpha^i)\,\frac{\sum_{sia}\,{}^3e^r_{(a)}\,{}^3e^s_{(a)}\,\partial_s\,\alpha^i\,\Pi_i}{\sqrt{\mu^2\,det^2\,(\partial_r\,\alpha^i) + \sum_{uvija}\,{}^3e^u_{(a)}\,{}^3e^v_{(a)}\,\partial_u\,\alpha^i\,\partial_v\,\alpha^j\,\Pi_i\,\Pi_j}},\\
J^r &= (1+n)\,Y^r - {}^3e^r_{(a)}\,n_{(a)}\,J^\tau =\\
&= det\,(\partial_r\,\alpha^i)\,\sum_a\,{}^3e^r_{(a)}\,\Big[\frac{(1+n)\,\sum_{si}\,{}^3e^s_{(a)}\,\partial_s\,\alpha^i\,\Pi_i}{\sqrt{\mu^2\,det^2\,(\partial_r\,\alpha^i) + \sum_{uvija}\,{}^3e^u_{(a)}\,{}^3e^v_{(a)}\,\partial_u\,\alpha^i\,\partial_v\,\alpha^j\,\Pi_i\,\Pi_j}} - n_{(a)}\Big],\\
J^\tau &= det\,(\partial_r\,\alpha^i).
\end{aligned}
\tag{4.6}
$$

Consequently, we can get the expression of the velocities of the Lagrangian coordinates in terms of the momenta by using Eq.(3.1) (whose validity in general relativity will be shown in Eq.(4.15))

$$
\begin{aligned}
\partial_\tau\,\alpha^i &= -\frac{\sum_r\,J^r\,\partial_r\,\alpha^i}{J^\tau} = \sum_r\,\frac{\Big(J^\tau\,\sum_a\,{}^3e^r_{(a)}\,n_{(a)} - (1+n)\,Y^r\Big)\,\partial_r\,\alpha^i}{J^\tau} =\\
&= \sum_{ra}\,\partial_r\,\alpha^i\,{}^3e^r_{(a)}\,\Big[n_{(a)} - \frac{(1+n)\,\sum_{si}\,{}^3e^s_{(a)}\,\partial_s\,\alpha^i\,\Pi_i}{\sqrt{\mu^2\,(J^\tau)^2 + \sum_{uvija}\,{}^3e^u_{(a)}\,{}^3e^v_{(a)}\,\partial_u\,\alpha^i\,\partial_v\,\alpha^j\,\Pi_i\,\Pi_j}}\Big],
\end{aligned}
\tag{4.7}
$$

so that the unit 4-velocity of the dust becomes (${}^4g_{AB}\,U^A\,U^B = \epsilon$; in the last line the 4-velocity is decomposed on the contra-variant ortho-normal tetrads $l^A(\tau,\vec{\sigma})$, ${}^4\mathop{\bar{E}}\limits^{\circ}{}^A_{(a)}(\tau,\vec{\sigma})$ carried by the Eulerian observers as shown in Eqs.(2.8) and (2.9))

$$
\begin{aligned}
U^\tau &= \frac{J^\tau}{(1+n)\,X} = \frac{1}{\mu\,(1+n)\,J^\tau}\,\sqrt{\mu^2\,(J^\tau)^2 + \sum_{uvija} {}^3e^u_{(a)}\,{}^3e^v_{(a)}\,\partial_u\,\alpha^i\,\partial_v\,\alpha^j\,\Pi_i\,\Pi_j},\\
U^r &= \frac{J^r}{(1+n)\,X} = \frac{\sum_a {}^3e^r_{(a)}}{\mu\,(1+n)\,J^\tau}\,\Big[(1+n)\,\sum_{si} {}^3e^s_{(a)}\,\partial_s\,\alpha^i\,\Pi_i -\\
&- n_{(a)}\,\sqrt{\mu^2\,(J^\tau)^2 + \sum_{uvija} {}^3e^u_{(a)}\,{}^3e^v_{(a)}\,\partial_u\,\alpha^i\,\partial_v\,\alpha^j\,\Pi_i\,\Pi_j}\Big],\\
&\\
U_\tau &= {}^4g_{\tau A}\,U^A = \frac{\epsilon\,(1+n)}{\mu\,J^\tau}\,\Big(\sqrt{\mu^2\,(J^\tau)^2 + \sum_{uvija} {}^3e^u_{(a)}\,{}^3e^v_{(a)}\,\partial_u\,\alpha^i\,\partial_v\,\alpha^j\,\Pi_i\,\Pi_j} -\\
&- \sum_{ari} \frac{n_{(a)}}{1+n}\,{}^3e^r_{(a)}\,\partial_r\,\alpha^i\,\Pi_i\Big),\\
U_r &= {}^4g_{rA}\,U^A = -\epsilon\,\Gamma(w)\,w_r = -\frac{1}{\mu\,J^\tau}\,\sum_i \partial_r\,\alpha^i\,\Pi_i,\\
U^A &= \frac{1}{2}\,(1+n)\,U^\tau\,\Big[l^A + \sum_a \frac{\bar{n}_{(a)} + 2\,\sum_r \frac{U^r}{U^\tau}\,{}^3\bar{e}_{(a)r}}{1+n}\,{}^4\mathring{\bar{E}}^A_{(a)}\Big].
\end{aligned}
\qquad (4.8)
$$

For the dust we have the following Hamiltonian expression of the energy-momentum tensor (3.9), of the energy density $\bar{\rho} = \epsilon\,\rho = \epsilon\,\mu\,\tilde{n}$ and of the mass density (3.11)

$$
\begin{aligned}
T^{AB} &= \epsilon\,\mu\,\tilde{n}\,U^A\,U^B \stackrel{def}{=} \bar{\rho}\,U^A\,U^B,\\
\mathcal{M} &= \sqrt{\gamma}\,(1+n)^2\,T^{\tau\tau} = (1+n)^2\,\sqrt{\gamma}\,\bar{\rho}\,(U^\tau)^2 = \sqrt{\mu^2\,\Big(J^\tau\Big)^2 + \sum_{uvija} {}^3e^u_{(a)}\,{}^3e^v_{(a)}\,\partial_u\,\alpha^i\,\partial_v\,\alpha^j\,\Pi_i\,\Pi_j},\\
\bar{\rho} &= \frac{T^{\tau\tau}}{(U^\tau)^2} = \frac{\mathcal{M}}{\sqrt{\gamma}\,(1+n)^2\,(U^\tau)^2},\\
\mathcal{M}_r &= \sqrt{\gamma}\,(1+n)\,\sum_s {}^3g_{rs}\,(T^{\tau s} + n^s\,T^{\tau\tau}) = \partial_r\,\alpha^i\,\Pi_i = -\mu\,J^\tau\,U_r,
\end{aligned}
$$

$$T^{rs} = \bar{\rho}\, U^r\, U^s =$$
$$= \frac{\epsilon}{(1+n)^2\sqrt{\gamma}} \frac{\sum_{ab} {}^3e^r_{(a)}\, {}^3e^s_{(b)}}{\sqrt{\mu^2 \Big(J^\tau\Big)^2 + \sum_{uvija} {}^3e^u_{(a)}\, {}^3e^v_{(a)}\, \partial_u\, \alpha^i\, \partial_v\, \alpha^j\, \Pi_i\, \Pi_j}}$$
$$\Big[(1+n) \sum_{ui} {}^3e^u_{(a)}\, \partial_u\, \alpha^i\, \Pi_i + n_{(a)} \sqrt{\mu^2 \Big(J^\tau\Big)^2 + \sum_{uvija} {}^3e^u_{(a)}\, {}^3e^v_{(a)}\, \partial_u\, \alpha^i\, \partial_v\, \alpha^j\, \Pi_i\, \Pi_j}\Big]$$
$$\Big[(1+n) \sum_{vj} {}^3e^v_{(b)}\, \partial_v\, \alpha^j\, \Pi_j + n_{(b)} \sqrt{\mu^2 \Big(J^\tau\Big)^2 + \sum_{uvija} {}^3e^u_{(a)}\, {}^3e^v_{(a)}\, \partial_u\, \alpha^i\, \partial_v\, \alpha^j\, \Pi_i\, \Pi_j}\Big]. \tag{4.9}$$

Let us remark that for the dust, having $p = 0$, Eqs.(2.9) and (3.19) imply $\mathcal{M} = \tilde{\phi}\, \Gamma^2(w)\, \bar{\rho} = \mu\, J^\tau\, \Gamma(w)$ and $\mathcal{M}_r = -\tilde{\phi}\Gamma^2(w)\, \bar{\rho}\, w_r = -\mathcal{M}\, w_r$.

The Hamilton equations for the dynamical matter, given in the next Subsection, imply the identities ${}^4\nabla_A\, T^{AB} \equiv 0$, consequence of Einstein equations due to the Bianchi identities, whose expression was given in Eqs.(3.18) or (3.20). For the dust Eqs.(3.20) become ($\rho = \epsilon\, \bar{\rho}$; $n^r = \sum_a \bar{n}_{(a)}\, {}^3\bar{e}^r_{(a)}$, ${}^3K_{rs} = \frac{1}{3}\, {}^3K\, {}^3g_{rs} + \sum_{ab} \sigma_{(a)(b)}\, {}^3\bar{e}_{(a)r}\, {}^3\bar{e}_{(b)s}$ with the notations of Appendix A)

$$\Big(\frac{\partial\, w_r}{\partial\, \tau} - \sum_s n^s\, \frac{\partial\, w_r}{\partial\, \sigma^s} - \sum_s w_s\, \frac{\partial\, n_s}{\partial\, \sigma^r}\Big) - (1+n) \sum_u w^u\, {}^3\nabla_u\, w_r - \frac{\partial\, n}{\partial\, \sigma^r} +$$
$$+ w_r \Big(\sum_u w^u\, \frac{\partial\, n}{\partial\, \sigma^u} + (1+n) \sum_{uv} w^u\, w^v\, {}^3K_{uv}\Big) \equiv 0,$$

$$\Big[\frac{\partial}{\partial\, \tau} - \sum_s \Big((1+n)\, w^s + n^s\Big)\, \frac{\partial}{\partial\, \sigma^s}\Big]\, \rho -$$
$$- \; (1+n)\, \rho \Big[\Big({}^3K - \sum_{rs} w^r\, w^s\, {}^3K_{rs}\Big) + \sum_r {}^3\nabla_r\, w^r\Big] \equiv 0. \tag{4.10}$$

At the Hamiltonian level they are a consequence of the Hamilton equations for the dust, given in Eqs.(4.14).

B. The Dust in the York Canonical Basis

In the York canonical basis (A2) we have ($\phi = \tilde{\phi}^{1/6} = (\sqrt{\gamma})^{1/6}$)

$\varphi_{(a)}$	$\alpha_{(a)}$	n	$\bar{n}_{(a)}$	θ^r	$\tilde{\phi}$	$R_{\bar{a}}$	α^i
$\pi_{\varphi_{(a)}} \approx 0$	$\pi^{(\alpha)}_{(a)} \approx 0$	$\pi_n \approx 0$	$\pi_{\bar{n}_{(a)}} \approx 0$	$\pi^{(\theta)}_r$	$\pi_{\tilde{\phi}}$	$\Pi_{\bar{a}}$	Π_i

$$
\begin{aligned}
X &= \tilde{\phi}\,\tilde{n} = \tilde{\phi}\,\frac{\rho}{\mu} = \frac{\mu\,(J^\tau)^2}{\sqrt{\mu^2\,(J^\tau)^2 + \tilde{\phi}^{-2/3}\,\sum_{auvij}\,Q_a^{-2}\,V_{ua}\,V_{va}\,\partial_u\,\alpha^i\,\partial_v\,\alpha^j\,\Pi_i\,\Pi_j}},\\
Y^r &= J^\tau\,\frac{\tilde{\phi}^{-2/3}\,\sum_{bsi}\,Q_b^{-2}\,V_{rb}\,V_{sb}\,\partial_s\,\alpha^i\,\Pi_i}{\sqrt{\mu^2\,(J^\tau)^2 + \tilde{\phi}^{-2/3}\,\sum_{auvij}\,Q_a^{-2}\,V_{ua}\,V_{va}\,\partial_u\,\alpha^i\,\partial_v\,\alpha^j\,\Pi_i\,\Pi_j}},\\
J^r &= (1+n)\,Y^r - \sum_a\,{}^3e^r_{(a)}\,n_{(a)}\,J^\tau = J^\tau\,\tilde{\phi}^{-1/3}\,\sum_b\,Q_b^{-1}\,V_{rb}\\
&\Big[\frac{(1+n)\,\tilde{\phi}^{-1/3}\,Q_b^{-1}\,\sum_{si}\,V_{sb}\,\partial_s\,\alpha^i\,\Pi_i}{\sqrt{\mu^2\,(J^\tau)^2 + \tilde{\phi}^{-2/3}\,\sum_{auvij}\,Q_a^{-2}\,V_{ua}\,V_{va}\,\partial_u\,\alpha^i\,\partial_v\,\alpha^j\,\Pi_i\,\Pi_j}} - n_{(b)}\Big],\\
J^\tau &= det\,(\partial_r\,\alpha^i),\\
\mathcal{T}_{ri} &= \frac{1}{2}\,\tilde{\phi}^{2/3}\,\sum_{asuvjk}\,Q_a^2\,V_{ra}\,V_{sa}\,\epsilon^{suv}\,\epsilon_{ijk}\,\partial_u\,\alpha^j\,\partial_v\,\alpha^k,\\
&\\
\partial_\tau\,\alpha^i &= -\frac{\sum_r\,J^r\,\partial_r\,\alpha^i}{J^\tau} = \frac{\sum_r\,\Big(\sum_a\,{}^3e^r_{(a)}\,n_{(a)}\,J^\tau - (1+n)\,Y^r\Big)\,\partial_r\,\alpha^i}{J^\tau} =\\
&= \tilde{\phi}^{-1/3}\,\sum_{rb}\,\partial_r\,\alpha^i\,Q_b^{-1}\,V_{rb}\,\Big[n_{(b)} - \frac{(1+n)\,\tilde{\phi}^{-1/3}\,Q_b^{-1}\,\sum_{si}\,V_{sb}\,\partial_s\,\alpha^i\,\Pi_i}{\sqrt{\mu^2\,(J^\tau)^2 + \tilde{\phi}^{-2/3}\,\sum_{auvij}\,Q_a^{-2}\,V_{ua}\,V_{va}\,\partial_u\,\alpha^i\,\partial_v\,\alpha^j\,\Pi_i\,\Pi_j}}\Big],\\
&\\
\mathcal{M} &= \sqrt{\mu^2\,(J^\tau)^2 + \tilde{\phi}^{-2/3}\,\sum_{arsij}\,Q_a^{-2}\,V_{ra}\,V_{sa}\,\partial_r\,\alpha^i\,\partial_s\,\alpha^j\,\Pi_i\,\Pi_j},\\
\mathcal{M}_r &= \sum_i\,\partial_r\,\alpha^i\,\Pi_i,\\
T^{rs} &= \frac{\epsilon\,\tilde{\phi}^{-5/3}}{(1+n)^2}\,\sum_{ab}\,\frac{Q_a^{-1}\,Q_b^{-1}\,V_{ra}\,V_{sb}}{\sqrt{\mu^2\,\Big(J^\tau\Big)^2 + \tilde{\phi}^{-2/3}\,\sum_{cuvij}\,Q_c^{-2}\,V_{uc}\,V_{vc}\,\partial_u\,\alpha^i\,\partial_v\,\alpha^j\,\Pi_i\,\Pi_j}}\\
&\Big[(1+n)\,\tilde{\phi}^{-1/3}\,Q_a^{-1}\,\sum_{mi}\,V_{ma}\,\partial_m\,\alpha^i\,\Pi_i + \bar{n}_{(a)}\,\sqrt{\mu^2\,\Big(J^\tau\Big)^2 + \tilde{\phi}^{-2/3}\,\sum_{cuvij}\,Q_c^{-2}\,V_{uc}\,V_{vc}\,\partial_u\,\alpha^i\,\partial_v\,\alpha^j\,\Pi_i\,\Pi_j}\Big]\\
&\Big[(1+n)\,\tilde{\phi}^{-1/3}\,Q_b^{-1}\,\sum_{nj}\,V_{nb}\,\partial_n\,\alpha^j\,\Pi_j + \bar{n}_{(b)}\,\sqrt{\mu^2\,\Big(J^\tau\Big)^2 + \tilde{\phi}^{-2/3}\,\sum_{cuvij}\,Q_c^{-2}\,V_{uc}\,V_{vc}\,\partial_u\,\alpha^i\,\partial_v\,\alpha^j\,\Pi_i\,\Pi_j}\Big].
\end{aligned}
\tag{4.11}
$$

The unit 4-velocity (4.6) and the energy density $\bar{\rho}$ of Eq.(4.9) of the dust are

$$
\begin{aligned}
U^\tau &= \frac{1}{\mu\,(1+n)\,J^\tau}\,\sqrt{\mu^2\,(J^\tau)^2 + \tilde{\phi}^{-2/3}\sum_{arsij} Q_a^{-2}\,V_{ra}\,V_{sa}\,\partial_r\,\alpha^i\,\partial_s\,\alpha^j\,\Pi_i\,\Pi_j},\\
U^r &= \frac{\tilde{\phi}^{-1/3}\,\sum_b Q_b^{-1}\,V_{rb}}{\mu\,(1+n)\,J^\tau}\,\Big[(1+n)\,\tilde{\phi}^{-1/3}\,Q_b^{-1}\sum_{si} V_{sb}\,\partial_s\,\alpha^i\,\Pi_i\\
&-\ \bar{n}_{(b)}\,\sqrt{\mu^2\,(J^\tau)^2 + \tilde{\phi}^{-2/3}\sum_{arsij} Q_a^{-2}\,V_{ra}\,V_{sa}\,\partial_r\,\alpha^i\,\partial_s\,\alpha^j\,\Pi_i\,\Pi_j}\Big],
\end{aligned}
$$

$$
\begin{aligned}
U_\tau &= {}^4g_{\tau\tau}\,U^\tau + {}^4g_{\tau s}\,U^s = \epsilon\,\Big([(1+n)^2 - \sum_a \bar{n}^2_{(a)}]\,U^\tau - \tilde{\phi}^{1/3}\sum_{as} Q_a\,\bar{n}_{(a)}\,V_{sa}\,U^s\Big) =\\
&= \frac{\epsilon\,(1+n)}{\mu\,J^\tau}\,\Big[\sqrt{\mu^2\,(J^\tau)^2 + \tilde{\phi}^{-2/3}\sum_{arsij} Q_a^{-2}\,V_{ra}\,V_{sa}\,\partial_r\,\alpha^i\,\partial_s\,\alpha^j\,\Pi_i\,\Pi_j} -\\
&-\ \tilde{\phi}^{-1/3}\sum_a \frac{\bar{n}_{(a)}}{1+n}\,Q_a^{-1}\sum_{si} V_{sa}\,\partial_s\,\alpha^i\,\Pi_i\Big],\\
U_r &= {}^4g_{r\tau}\,U^\tau + {}^4g_{rs}\,U^s = -\epsilon\,\tilde{\phi}^{1/3}\sum_a V_{ra}\,Q_a\,\Big(\bar{n}_{(a)}\,U^\tau + \tilde{\phi}^{1/3}\,Q_a\sum_s V_{sa}\,U^s\Big) =\\
&= -\frac{1}{\mu\,J^\tau}\sum_i \partial_r\,\alpha^i\,\Pi_i = -\frac{\mathcal{M}_r}{\mu\,J^\tau},\\
\bar{\rho} &= \frac{\mu^2\,(J^\tau)^2\,\tilde{\phi}^{-1}}{\sqrt{\mu^2\,(J^\tau)^2 + \tilde{\phi}^{-2/3}\sum_{arsij} Q_a^{-2}\,V_{ra}\,V_{sa}\,\partial_r\,\alpha^i\,\partial_s\,\alpha^j\,\Pi_i\,\Pi_j}}.
\end{aligned}
\tag{4.12}
$$

C. The Hamilton Equations in the York Canonical Basis

The Hamilton equations of the gravitational field in Schwinger time gauges are generated by the following Dirac Hamiltonian (see Eq.(3.48) of Ref. [2] for H_{grav}), which depends upon matter only through the mass density and the mass current density

$$
\begin{aligned}
H_D &= \frac{1}{c}\,\hat{E}_{ADM} + \int d^3\sigma\,\Big[n\,\mathcal{H} - \bar{n}_{(a)}\,\bar{\tilde{\mathcal{H}}}_{(a)}\Big](\tau,\vec{\sigma}) +\\
&+\ \int d^3\sigma\,\Big[\lambda_n\,\tilde{\pi}^n + \lambda^{\vec{n}}_{(a)}\,\tilde{\pi}^{\vec{n}}_{(a)}\Big](\tau,\vec{\sigma}) =\\
&=\ H_{grav} + \int d^3\sigma\,\Big[(1+n)\,\mathcal{M}\Big](\tau,\vec{\sigma}) - \int d^3\sigma\,\sum_r n^r(\tau,\vec{\sigma})\,\mathcal{M}_r(\tau,\vec{\sigma}) =\\
&=\ H_{grav} + \int d^3\sigma\,\Big[(1+n)\,\mathcal{M}\Big](\tau,\vec{\sigma}) -\\
&-\ \int d^3\sigma\,\sum_a \Big(\bar{n}_{(a)}\ \tilde{\phi}^{-1/3}\,Q_a^{-1}\sum_r V_{ra}(\theta^i)\,\mathcal{M}_r\Big)(\tau,\vec{\sigma}).
\end{aligned}
\tag{4.13}
$$

and are explicitly given in Section IV of Ref.[2]. The expressions of the super-Hamiltonian and super-momentum constraints in the York canonical basis are given in Eqs. (3.45) and (3.42) of Ref.[2].

The Hamilton equations for the dust are

$$
\begin{aligned}
\partial_\tau\, \alpha^i(\tau,\vec\sigma) &\stackrel{\circ}{=} -\int d^3\sigma_1\, [1+n(\tau,\vec\sigma_1)]\, \frac{\delta \mathcal{M}(\tau,\vec\sigma_1)}{\delta\, \Pi_i(\tau,\vec\sigma)} + \\
&+ \int d^3\sigma_1\, [\tilde\phi^{-1/3} \sum_a \bar n_{(a)}\, Q_a^{-1} \sum_r V_{ra}](\tau,\vec\sigma_1)\, \frac{\delta\, \mathcal{M}_r(\tau,\vec\sigma_1)}{\delta\, \Pi_i(\tau,\vec\sigma)} = \\
&= -\Big((1+n)\, \tilde\phi^{-2/3}\, \frac{\sum_{arsj}\, Q_a^{-2}\, V_{ra}\, V_{sa}\, \partial_r\, \alpha^i\, \partial_s\, \alpha^j\, \Pi_j}{\sqrt{\mu^2\, (J^\tau)^2 + \tilde\phi^{-2/3}\, \sum_{buvmn}\, Q_b^{-2}\, V_{ub}\, V_{vb}\, \partial_u\, \alpha^m\, \partial_v\, \alpha^n\, \Pi_m\, \Pi_n}} - \\
&- \tilde\phi^{-1/3} \sum_a \bar n_{(a)}\, Q_a^{-1} \sum_r V_{ra}\, \partial_r\, \alpha^i\Big)(\tau,\vec\sigma),\\
\partial_\tau\, \Pi_i(\tau,\vec\sigma) &\stackrel{\circ}{=} \int d^3\sigma_1\, [1+n(\tau,\vec\sigma_1)]\, \frac{\delta \mathcal{M}(\tau,\vec\sigma_1)}{\delta\, \alpha^i(\tau,\vec\sigma)} - \\
&- \int d^3\sigma_1\, [\tilde\phi^{-1/3} \sum_a \bar n_{(a)}\, Q_a^{-1} \sum_r V_{ra}](\tau,\vec\sigma_1)\, \frac{\delta\, \mathcal{M}_r(\tau,\vec\sigma_1)}{\delta\, \alpha^i(\tau,\vec\sigma)} =
\end{aligned}
$$

$$
\begin{aligned}
&= -\sum_r \partial_r\, \Big(\frac{1+n}{\sqrt{\mu^2\, (J^\tau)^2 + \tilde\phi^{-2/3}\, \sum_{buvmn}\, Q_b^{-2}\, V_{ub}\, V_{vb}\, \partial_u\, \alpha^m\, \partial_v\, \alpha^n\, \Pi_m\, \Pi_n}}\\
&\Big[\frac{1}{2}\, \mu^2\, J^\tau \sum_{uvjk} \epsilon^{ruv}\, \epsilon_{ijk}\, \partial_u\, \alpha^j\, \partial_v\, \alpha^k + \tilde\phi^{-2/3} \sum_{asj} Q_a^{-2}\, V_{ra}\, V_{sa}\, \partial_s\, \alpha^j\, \Pi_j\, \Pi_i\Big] - \\
&- \tilde\phi^{-1/3} \sum_a \bar n_{(a)}\, Q_a^{-1}\, \Pi_i\Big)(\tau,\vec\sigma).
\end{aligned}
\tag{4.14}
$$

The first half of these equations allows to check the validity at the Hamiltonian level of the basic defining property of the Lagrangian (comoving) scalar fields $\alpha^i(\tau,\vec\sigma)$ used as coordinates for the fluid flux lines

$$
U^A(\tau,\vec\sigma)\, {}^4\nabla_A\, \alpha^i(\tau,\vec\sigma) = U^A(\tau,\vec\sigma)\, \partial_A\, \alpha^i(\tau,\vec\sigma) \stackrel{\circ}{=} 0. \tag{4.15}
$$

In accord with Eq.(4.2) the inversion of the first equation gives ($\mathcal{I}_{ir} = \partial_r\, \alpha^i$ from Eq.(3.6))

$$\Pi_i = \frac{\mu\, J^\tau\, \tilde{\phi}^{2/3}\, \sum_{rscj}\, \mathcal{I}^{-1}_{ir}\, Q_c^2\, V_{rc}\, V_{sc}\, \mathcal{I}^{-1}_{js}\, [\partial_\tau\, \alpha^j - \tilde{\phi}^{-1/3}\, \sum_{au}\, \bar{n}_{(a)}\, Q_a^{-1}\, V_{ua}\, \partial_u\, \alpha^j]}{\sqrt{(1+n)^2 - \tilde{\phi}^{2/3}\, \sum_b\, \Big[Q_b\, \sum_{sk}\, V_{sb}\, \mathcal{I}^{-1}_{ks}\, [\partial_\tau\, \alpha^k - \tilde{\phi}^{-1/3}\, \sum_{ev}\, \bar{n}_{(e)}\, Q_e^{-1}\, V_{ve}\, \partial_v\, \alpha^k]\Big]^2}},$$

$$\mathcal{I}^{-1}_{ir} = \frac{1}{2\, J^\tau}\, \epsilon^{ruv}\, \epsilon_{ijk}\, \partial_u\, \alpha^j\, \partial_v\, \alpha^k,$$

$$\frac{1}{\sqrt{\mu^2\, (J^\tau)^2 + \tilde{\phi}^{-2/3}\, \sum_{buv}\, Q_b^{-2}\, V_{ub}\, V_{vb}\, \partial_u\, \alpha^m\, \partial_v\, \alpha^n\, \Pi_m\, \Pi_n}} = \frac{\sqrt{(1+n)^2 - \tilde{\phi}^{2/3}\, \sum_b\, \Big[Q_b\, \sum_{sk}\, V_{sb}\, \mathcal{I}^{-1}_{ks}\, [\partial_\tau\, \alpha^k - \tilde{\phi}^{-1/3}\, \sum_{ev}\, \bar{n}_{(e)}\, Q_e^{-1}\, V_{ve}\, \partial_v\, \alpha^k]\Big]^2}}{\mu\, J^\tau\, (1+n)}. \tag{4.16}$$

By putting this expression for $\Pi_i(\tau, \vec{\sigma})$ in the second set of Hamilton equations, we can get the second order equations of motion for the dust variables $\alpha^i(\tau, \vec{\sigma})$: they have the form of three conserved currents $\partial_A\, j_i^A(\tau, \vec{\sigma}) \circeq 0$. These conservation laws imply that the quantities $\int d^3\sigma\, \Pi_i(\tau, \vec{\sigma})$ are constant of the motion and generate the transformation $\alpha^i(\tau, \vec{\sigma}) \rightarrow \alpha^i(\tau, \vec{\sigma}) + const.$ leaving invariant the Lagrangian (4.1).

Finally the Hamilton-Dirac Equation (4.15) for the dust are equivalent to the 4-dimensional, manifestly covariant, equations

$$U^A\, {}^4\nabla_A\, U_B \circeq 0, \tag{4.17}$$

implying that the flux lines are geodesics. For the dust with null pressure Eq.(4.17) is equivalent to eq.(3.16).

D. The 3-Orthogonal Schwinger Time Gauges

The restriction of the Hamilton equations to the family of (non-harmonic) 3-orthogonal Schwinger time gauges, where $\theta^i(\tau, \vec{\sigma}) \approx 0$ so that the 3-metric is diagonal (${}^3g_{rs} = \tilde{\phi}^{2/3}\, Q_r^2\, \delta_{rs}$ from Eq.(A3)), is given in Section II of Ref.[3]. The restriction to these gauges of Eqs.(4.14) is

$$
\begin{aligned}
\mathcal{M} &= \sqrt{\mu^2\,(J^\tau)^2 + \tilde{\phi}^{-2/3}\,\sum_a Q_a^{-2}\,(\sum_i\,\partial_a\,\alpha^i\,\Pi_i)^2},\\
\mathcal{M}_r &= \sum_i \partial_r\,\alpha^i\,\Pi_i,\\
T^{rs} &= \frac{\epsilon\,\tilde{\phi}^{-5/3}}{(1+n)^2}\,\frac{Q_r^{-1}\,Q_s^{-1}}{\sqrt{\mu^2\,\Big(J^\tau\Big)^2 + \tilde{\phi}^{-2/3}\,\sum_c Q_c^{-2}\,(\sum_i\,\partial_c\,\alpha^i\,\Pi_i)^2}}\\
&\Big[(1+n)\,\tilde{\phi}^{-1/3}\,Q_a^{-1}\,\sum_i\,\partial_a\,\alpha^i\,\Pi_i + \bar{n}_{(a)}\,\sqrt{\mu^2\,\Big(J^\tau\Big)^2 + \tilde{\phi}^{-2/3}\,\sum_c\,Q_c^{-2}\,(\sum_i\,\partial_c\,\alpha^i\,\Pi_i)^2}\Big]\\
&\Big[(1+n)\,\tilde{\phi}^{-1/3}\,Q_b^{-1}\,\sum_j\,\partial_b\,\alpha^j\,\Pi_j + \bar{n}_{(b)}\,\sqrt{\mu^2\,\Big(J^\tau\Big)^2 + \tilde{\phi}^{-2/3}\,\sum_c\,Q_c^{-2}\,(\sum_j\,\partial_c\,\alpha^j\,\Pi_j)^2}\Big],\\
U^\tau &= \frac{1}{\mu\,(1+n)\,J^\tau}\,\sqrt{\mu^2\,\Big(J^\tau\Big)^2 + \tilde{\phi}^{-2/3}\,\sum_c\,Q_c^{-2}\,(\sum_i\,\partial_c\,\alpha^i\,\Pi_i)^2},\\
U^r &= \frac{\tilde{\phi}^{-1/3}\,Q_r^{-1}}{\mu\,(1+n)\,J^\tau}\,\Big[(1+n)\,\tilde{\phi}^{-1/3}\,\sum_i\,Q_r^{-1}\,\partial_r\,\alpha^i\,\Pi_i -\\
&- \bar{n}_{(r)}\,\sqrt{\mu^2\,(J^\tau)^2 + \tilde{\phi}^{-2/3}\,\sum_a\,Q_a^{-2}\,(\sum_i\,\partial_a\,\alpha^i\,\Pi_i)^2},
\end{aligned}
$$

$$
\begin{aligned}
U_\tau &= \frac{\epsilon\,(1+n)}{\mu\,J^\tau}\,\Big[\sqrt{\mu^2\,(J^\tau)^2 + \tilde{\phi}^{-2/3}\,\sum_{aij}\,Q_a^{-2}\,\partial_a\,\alpha^i\,\partial_a\,\alpha^j\,\Pi_i\,\Pi_j} -\\
&- \tilde{\phi}^{-1/3}\,\sum_a\,\frac{\bar{n}_{(a)}}{1+n}\,Q_a^{-1}\,\sum_i\,\partial_a\,\alpha^i\,\Pi_i\Big],\\
U_r &= -\epsilon\,\Gamma(w)\,w_r = -\frac{1}{\mu\,J^\tau}\,\sum_i\,\partial_r\,\alpha^i\,\Pi_i,
\end{aligned}
$$

$$
\begin{aligned}
\partial_\tau\, \alpha^i(\tau,\vec{\sigma}) \,&\stackrel{\circ}{=}\, -\Big((1+n)\, \tilde{\phi}^{-2/3}\, \frac{\sum_{aj}\, Q_a^{-2}\, \partial_a\, \alpha^i\, \partial_a\, \alpha^j\, \Pi_j}{\sqrt{\mu^2\, (J^\tau)^2 + \tilde{\phi}^{-2/3}\, \sum_{bmn}\, Q_b^{-2}\, \partial_b\, \alpha^m\, \partial_b\, \alpha^n\, \Pi_m\, \Pi_n}} - \\
&-\, \tilde{\phi}^{-1/3}\, \sum_a\, \bar{n}_{(a)}\, Q_a^{-1}\, \partial_a\, \alpha^i\Big)(\tau,\vec{\sigma}),\\
\Pi_i \,&=\, \frac{\mu\, J^\tau\, \tilde{\phi}^{2/3}\, \sum_{cj}\, \mathcal{I}^{-1}_{ic}\, Q_c^2\, \mathcal{I}^{-1}_{jc}\, [\partial_\tau\, \alpha^j - \tilde{\phi}^{-1/3}\, \sum_a\, \bar{n}_{(a)}\, Q_a^{-1}\, \partial_a\, \alpha^j]}{\sqrt{(1+n)^2 - \tilde{\phi}^{2/3}\, \sum_b\, \Big[Q_b\, \sum_k\, \mathcal{I}^{-1}_{kb}\, [\partial_\tau\, \alpha^k - \tilde{\phi}^{-1/3}\, \sum_e\, \bar{n}_{(e)}\, Q_e^{-1}\, \partial_e\, \alpha^k]\Big]^2}},\\
\partial_\tau\, \Pi_i(\tau,\vec{\sigma}) \,&\stackrel{\circ}{=}\, -\sum_r\, \partial_r\, \Big(\frac{1+n}{\sqrt{\mu^2\, (J^\tau)^2 + \tilde{\phi}^{-2/3}\, \sum_{bmn}\, Q_b^{-2}\, \partial_b\, \alpha^m\, \partial_b\, \alpha^n\, \Pi_m\, \Pi_n}}\\
&\Big[\frac{1}{2}\, \mu^2\, J^\tau\, \sum_{uvjk}\, \epsilon^{ruv}\, \epsilon_{ijk}\, \partial_u\, \alpha^j\, \partial_v\, \alpha^k + \tilde{\phi}^{-2/3}\, Q_r^{-2}\, \sum_j\, \partial_r\, \alpha^j\, \Pi_j\, \Pi_i\Big] - \\
&-\, \tilde{\phi}^{-1/3}\, \sum_a\, \bar{n}_{(a)}\, Q_a^{-1}\, \Pi_i\Big)(\tau,\vec{\sigma}).
\end{aligned}
\tag{4.18}
$$

E. The Hamiltonian Post-Minkowskian Linearization

We now remember the results of Refs.[3, 4] on the Hamiltonian Post-Minkowskian linearization of ADM tetrad gravity with dynamical matter in the 3-orthogonal Schwinger time gauges by using the dust as matter. The asymptotic Minkowski 4-metric at spatial infinity is used as an *asymptotic background*. Post-Newtonian expansions are avoided by introducing a ultraviolet cutoff $M\, c^2$ for the energy of the matter.

The basic assumption is that on each instantaneous 3-space Σ_τ we have the following limitation of the a-dimensional configurational tidal variables $R_{\bar{a}}$ in the York canonical basis

$$
\begin{aligned}
&|R_{\bar{a}}(\tau,\vec{\sigma}) = R_{(1)\bar{a}}(\tau,\vec{\sigma})| = O(\zeta) << 1,\\
&|\partial_u\, R_{\bar{a}}(\tau,\vec{\sigma})| \sim \frac{1}{L} O(\zeta), \qquad |\partial_u\, \partial_v\, R_{\bar{a}}(\tau,\vec{\sigma})| \sim \frac{1}{L^2} O(\zeta),\\
&|\partial_\tau\, R_{\bar{a}}| = \frac{1}{L}\, O(\zeta), \qquad |\partial_\tau^2\, R_{\bar{a}}| = \frac{1}{L^2}\, O(\zeta), \qquad |\partial_\tau\, \partial_u\, R_{\bar{a}}| = \frac{1}{L^2}\, O(\zeta),\\
&\Rightarrow\;\; Q_a(\tau,\vec{\sigma}) = e^{\sum_{\bar{a}}\, \gamma_{\bar{a}a}\, R_{\bar{a}}(\tau,\vec{\sigma})} = 1 + \Gamma_a^{(1)}(\tau,\vec{\sigma}) + O(\zeta^2),\\
&\qquad \Gamma_a^{(1)} = \sum_{\bar{a}}\, \gamma_{\bar{a}a}\, R_{\bar{a}}, \qquad \sum_a\, \Gamma_a^{(1)} = 0, \qquad R_{\bar{a}} = \sum_a\, \gamma_{\bar{a}a}\, \Gamma_a^{(1)},
\end{aligned}
\tag{4.19}
$$

where L is a *big enough characteristic length interpretable as the reduced wavelength $\lambda/2\pi$ of the resulting GW's*. Therefore the tidal variables $R_{\bar{a}}$ are slowly varying over the length L and times L/c. This also implies that the Riemann tensor ${}^4R_{ABCD}$, the Ricci tensor ${}^4R_{AB}$ and the scalar 4-curvature ${}^4R$ behave as $\frac{1}{L^2}\, O(\zeta)$. Also the intrinsic 3-curvature scalar of the instantaneous 3-spaces Σ_τ, given in Eqs.(2.5), is of order $\frac{1}{L^2}\, O(\zeta)$. To simplify the notation we use $R_{\bar a}$ for $R_{(1)\bar a}$ in the rest of the paper.

For the other gravitational variables we make the following consistent assumptions (the notation $f_{(k)} = O(\zeta^k)$ is used)

$$
\begin{aligned}
\tilde \phi &= 1 + 6\, \phi_{(1)} + O(\zeta^2),\\
N &= 1 + n = 1 + n_{(1)} + O(\zeta^2),\\
\bar n_{(a)} &= \bar n_{(1)(a)} + O(\zeta^2),\\
&\Downarrow\\
{}^4g_{\tau\tau} &= \epsilon\, \Big(1 + 2\, n_{(1)}\Big) + O(\zeta^2),\\
{}^4g_{\tau r} &= -\epsilon\, \bar n_{(1)(r)} + O(\zeta^2),\\
{}^4g_{rs} &= -\epsilon\, {}^3g_{rs} = -\epsilon\, \Big(1 + 2\, (\Gamma_r^{(1)} + 2\, \phi_{(1)})\Big)\, \delta_{rs} + O(\zeta^2).
\end{aligned} \tag{4.20}
$$

Moreover we have (see Appendix A for the shear $\sigma_{(a)(b)}$ of the Eulerian observers)

$$
\begin{aligned}
\frac{8\pi\, G}{c^3}\, \Pi_{\bar a}(\tau, \vec \sigma) &= \frac{8\pi\, G}{c^3}\, \Pi_{(1)\bar a}(\tau, \vec \sigma) = \frac{1}{L}\, O(\zeta) \circeq \Big[\partial_\tau\, R_{\bar a} - \sum_a\, \gamma_{\bar aa}\, \partial_a\, \bar n_{(1)(a)}\Big](\tau, \vec \sigma) + \frac{1}{L}\, O(\zeta^2),\\
&\sigma_{(a)(a)} = \sigma_{(1)(a)(a)} = -\frac{8\pi\, G}{c^3}\, \sum_{\bar a}\, \gamma_{\bar aa}\, \Pi_{(1)\bar a} + \frac{1}{L}\, O(\zeta^2),\\
&\sigma_{(a)(b)}|_{a \not= b} = \sigma_{(1)(a)(b)}|_{a \not= b} = \frac{1}{L}\, O(\zeta),\\
&\Rightarrow \quad \frac{8\pi\, G}{c^3}\, \pi_i^{(\theta)} = \frac{1}{L}\, O(\zeta^2) = \sum_{a \not= b}\, (\Gamma_a^{(1)} - \Gamma_b^{(1)})\, \epsilon_{iab}\, \sigma_{(1)(a)(b)} + \frac{1}{L}\, O(\zeta^3),\\
&{}^3K = \frac{12\pi\, G}{c^3}\, \pi_{\tilde \phi} = {}^3K_{(1)} = \frac{12\pi\, G}{c^3}\, \pi_{(1)\tilde \phi} = \frac{1}{L}\, O(\zeta),\\
&\Downarrow\\
{}^3K_{rs} &= {}^3K_{(1)rs} = \frac{1}{L}\, O(\zeta) =\\
&= (1 - \delta_{rs})\, \sigma_{(1)(r)(s)} + \delta_{rs}\, \Big[\frac{1}{3}\, {}^3K_{(1)} - \partial_\tau\, \Gamma_r^{(1)} + \sum_a\, (\delta_{ra} - \frac{1}{3})\, \partial_a\, \bar n_{(1)(a)}\Big] + \frac{1}{L}\, O(\zeta^2).
\end{aligned} \tag{4.21}
$$

For the matter we must have

$$\mathcal{M}(\tau,\vec{\sigma}) = \mathcal{M}_{(1)}(\tau,\vec{\sigma}) + \mathcal{R}_{(2)}(\tau,\vec{\sigma}),$$

$$m_i = M\,O(\zeta), \qquad \int d^3\sigma\, \mathcal{M}_{(1)}(\tau,\vec{\sigma}) = Mc\,O(\zeta), \qquad \int d^3\sigma\, \mathcal{R}_{(2)}(\tau,\vec{\sigma}) = Mc\,O(\zeta^2),$$

$$\mathcal{M}_r(\tau,\vec{\sigma}) = \mathcal{M}_{(1)r}(\tau,\vec{\sigma}), \qquad \int d^3\sigma\, \mathcal{M}_{(1)r}(\tau,\vec{\sigma}) = Mc\,O(\zeta), \tag{4.22}$$

where M is the ultraviolet cutoff. We have $\mathcal{M}_{(1)}(\tau,\vec{\sigma}) = \frac{Mc}{L^3}\,O(\zeta)$, where the length L is the wavelength of gravitational waves as shown in Ref.[3].

For the dust we have $\alpha^i(\tau,\vec{\sigma}) = O(1)$ and that the mass scale $m = \frac{\mu}{c}$ must satisfy $m = M\,O(\zeta)$. Therefore the dust momenta are first order quantities $\Pi_i(\tau,\vec{\sigma}) = \Pi_{(1)i}(\tau,\vec{\sigma})$ such that $\int d^3\sigma\, \mathcal{M}_{(1)r}(\tau,\vec{\sigma}) = \int d^3\sigma\, \Big(\sum_i \partial_r\, \alpha^i\, \Pi_i\Big)(\tau,\vec{\sigma}) = Mc\,O(\zeta)$. These conditions also imply $\int d^3\sigma\, \mathcal{M}_{(1)}(\tau,\vec{\sigma}) = Mc\,O(\zeta)$.

As a consequence we have the following results for the energy-momentum tensor and for the 4-velocity of the dust (see Refs. [3, 4] for the expression of the generators of the asymptotic ADM Poincare' group)

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}_{(1)} + \frac{Mc}{l^3}\,O(\zeta^2),\\
&\quad \mathcal{M}_{(1)} = \sqrt{\mu^2\,(J^\tau)^2 + \sum_a (\sum_i \partial_a\,\alpha^i\,\Pi_i)^2},\\
\mathcal{M}_r &= \mathcal{M}_{(1)r} = -\partial_r\,\alpha^i\,\Pi_i + \frac{Mc}{L^3}\,O(\zeta^2),\\
T^{rs}_{(1)} &= -\frac{\epsilon\,\sum_{ij}\partial_r\,\alpha^i\,\Pi_i\,\partial_s\,\alpha^j\,\Pi_j}{\sqrt{\mu^2\,\Big(J^\tau\Big)^2 + \sum_c(\sum_i \partial_c\,\alpha^i\,\Pi_i)^2}} + \frac{Mc}{L^3}\,O(\zeta^2),\\
&\quad \partial_\tau\,\mathcal{M}_{(1)} + \partial_r\,\mathcal{M}_{(1)r} = 0 + \frac{Mc}{L^4}\,O(\zeta^2),\\
&\quad \partial_\tau\,\mathcal{M}_{(1)r} + \partial_s\,T^{rs}_{(1)} = 0 + \frac{Mc}{L^4}\,O(\zeta^2),\\
U^\tau &= \frac{1}{\mu\,J^\tau}\,\sqrt{\mu^2\,\Big(J^\tau\Big)^2 + \sum_c(\sum_i \partial_c\,\alpha^i\,\Pi_i)^2} + O(\zeta),\\
U^r &= -\frac{1}{\mu\,J^\tau}\,\partial_r\,\alpha^i\,\Pi_i + O(\zeta).
\end{aligned} \tag{4.23}$$

From Refs.[3, 4] we have the following solutions of the linearized equations for the gravitational field

$$
\begin{aligned}
\phi_{(1)}(\tau,\vec{\sigma}) \;&\circeq\; \Big[-\frac{2\pi\, G}{c^3}\,\frac{1}{\triangle}\,\mathcal{M}_{(1)} + \frac{1}{4}\,\sum_c \frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}\Big](\tau,\vec{\sigma}),\\
n_{(1)}(\tau,\vec{\sigma}) \;&\circeq\; \Big[\frac{4\pi\, G}{c^3}\,\frac{1}{\triangle}\,\Big(\mathcal{M}_{(1)} + \sum_a T^{aa}_{(1)}\Big) - \frac{1}{\triangle}\,\partial_\tau\, {}^3K_{(1)}\Big](\tau,\vec{\sigma}),\\
\bar{n}_{(1)(a)}(\tau,\vec{\sigma}) \;&\circeq\; \Big[\frac{\partial_a}{\triangle}\,{}^3K_{(1)} + \frac{4\pi\, G}{c^3}\,\frac{1}{\triangle}\,\Big(4\,\mathcal{M}_{(1)a} - \frac{\partial_a}{\triangle}\,\sum_c \partial_c\,\mathcal{M}_{(1)c}\Big) + \\
&+\; \frac{1}{2}\,\partial_\tau\,\frac{\partial_a}{\triangle}\,\Big(4\,\Gamma_a^{(1)} - \sum_c \frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}\Big)\Big](\tau,\vec{\sigma}),\\
\sigma_{(1)(a)(b)}|_{a\neq b} \;&\circeq\; \frac{1}{2}\,\Big(\partial_a\,\bar{n}_{(1)(b)} + \partial_b\,\bar{n}_{(1)(a)}\Big)(\tau,\vec{\sigma}).
\end{aligned}
\qquad (4.24)
$$

The retarded solution for the tidal variables (the gravitational waves) and for the TT 3-metric are (a multipolar expansion has been used; $M_{\bar{a}\bar{b}} = \delta_{\bar{a}\bar{b}} - \sum_a \gamma_{\bar{a}a}\,\frac{\partial_a^2}{\triangle}\Big(2\,\gamma_{\bar{b}a} - \frac{1}{2}\,\sum_b \gamma_{\bar{b}b}\,\frac{\partial_b^2}{\triangle}\Big)$; $q^{uv|\tau\tau}$ is the quadrupole; see Ref.[3] for the tensors Λ_{rsuv} and $\mathcal{P}_{rsuv}$)

$$
\begin{aligned}
{}^4h^{TT}_{(1)rs}(\tau,\vec{\sigma}) \;&\circeq\; -\epsilon\,\frac{4\,G}{c^3}\,\int d^3\sigma_1\,d^3\sigma_2\,\sum_{uv} d^{TT}_{rsuv}(\vec{\sigma}_1-\vec{\sigma}_2)\,\frac{T^{uv}_{(1)}(\tau-|\vec{\sigma}-\vec{\sigma}_1|,\vec{\sigma}_2)}{|\vec{\sigma}-\vec{\sigma}_1|} =\\
&=\; -\epsilon\,\frac{2\,G}{c^3}\,\sum_{uv}\Lambda_{rsuv}(n)\,\frac{\partial_\tau^2\,q^{uv|\tau\tau}((\tau-|\vec{\sigma}|))}{|\vec{\sigma}|} + (higher\,multipoles) + O(1/r^2),
\end{aligned}
$$

$$
\begin{aligned}
R_{\bar{a}}(\tau,\vec{\sigma}) \;&=\; \sum_a \gamma_{\bar{a}a}\,\Gamma_a^{(1)}(\tau,\vec{\sigma}) \circeq \qquad [\Gamma_a^{(1)}(\tau,\vec{\sigma}) = \sum_{\bar{a}} \gamma_{\bar{a}a}\,R_{\bar{a}}(\tau,\vec{\sigma})]\\
&\circeq\; -\frac{2\,G}{c^2}\,\sum_{ab}\gamma_{\bar{a}a}\,\tilde{M}^{-1}_{ab}(\vec{\sigma})\,\int d^3\sigma_1\,\int d^3\sigma_2\,\sum_{uv} d^{TT}_{bbuv}(\vec{\sigma}_1-\vec{\sigma}_2)\\
&\quad \frac{T^{uv}_{(1)}(\tau-|\vec{\sigma}-\vec{\sigma}_1|,\vec{\sigma}_2)}{|\vec{\sigma}-\vec{\sigma}_1|} + O(\zeta^2) =\\
&=\; -\frac{G}{c^3}\,\sum_{ab}\gamma_{\bar{a}a}\,\tilde{M}^{-1}_{ab}(\vec{\sigma})\,\frac{\sum_{uv}\mathcal{P}_{bbuv}\,\partial_\tau^2\,q^{uv|\tau\tau}(\tau-|\vec{\sigma}|)}{|\vec{\sigma}|} + (higher\,multipoles) + O(1/r^2),
\end{aligned}
$$

$$
q^{uv|\tau\tau}(\tau-|\vec{\sigma}|) \;=\; \int d^3\sigma_1\,\sigma_1^u\,\sigma_1^v\,\mathcal{M}_{(1)}(\tau-|\vec{\sigma}|,\vec{\sigma}_1). \qquad (4.25)
$$

The tidal momenta can be obtained from Eq.(4.21)

The linearization of Eqs.(4.18) is

$$
\begin{aligned}
\partial_\tau\, \alpha^i \;&=\; \sum_a \Big(1 + n_{(1)} - 4\,\phi_{(1)} - 2\,\Gamma^{(1)}_a\Big)\, \frac{\partial_a\, \alpha^i\, \partial_a\, \alpha^j\, \Pi_j}{\sqrt{\mu^2\, (J^\tau)^2 + \sum_b (\partial_b\, \alpha^k\, \Pi_k)^2}} + \sum_a \partial_a\, \alpha^i\, \bar{n}_{(1)(a)},\\
\partial_\tau\, \Pi_i \;&\stackrel{\circ}{=}\; \sum_r \partial_r \Big(\frac{1}{\sqrt{\mu^2\, (J^\tau)^2 + \sum_b (\partial_b\, \alpha^k\, \Pi_k)^2}} \Big[\frac{1}{2}\, \mu^2\, J^\tau\, \epsilon^{ruv}\, \epsilon_{ijk}\, \partial_u\, \alpha^j\, \partial_v\, \alpha^k + \sum_j \partial_r\, \alpha^j\, \Pi_j\, \Pi_i\Big] -\\
&- \sum_a \bar{n}_{(1)(a)}\, \Pi_i\Big),\\
\Pi_i \;&=\; \frac{\mu\, J^\tau}{\sqrt{1 - \sum_b (\sum_j \mathcal{I}^{-1}_{jb}\, \partial_\tau\, \alpha^j)^2}} \sum_{cj} \mathcal{I}^{-1}_{ic}\, \mathcal{I}^{-1}_{jc}\, \partial_\tau\, \alpha^j + O(\zeta^2),\;\; \text{since}\, \mu = mc = Mc\, O(\zeta).
\end{aligned}
\tag{4.26}
$$

By putting the solution (4.25) for the gravitational waves into these equations we get an integral differential equation for the dust.

The HPM linerization of the Bianchi identities (4.10) is ($w_r = \mathcal{M}_{(1)r}/\mathcal{M}_{(1)} = O(1)$)

$$
\begin{aligned}
&\Big(\frac{\partial\, w_r}{\partial\, \tau} - \sum_s \bar{n}_{(1)(s)}\, \frac{\partial\, w_r}{\partial\, \sigma^s} - \sum_s w_s\, \frac{\partial\, \bar{n}_{(1)(s)}}{\partial\, \sigma^r}\Big) -\\
&-(1 + n_{(1)}) \sum_u w^u\, \partial_u\, w_r - 2 \sum_u (w_u)^2\, \partial_r\, \phi_{(1)} - \sum_s (w_s)^2\, \partial_r\, \Gamma^{(1)}_s +\\
&\qquad - \frac{\partial\, n_{(1)}}{\partial\, \sigma^r} + w_r \Big(\sum_u w^u\, \frac{\partial\, n_{(1)}}{\partial\, \sigma^u} + (1 + n) \sum_{uv} w^u\, w^v\, {}^3K_{(1)uv}\Big) \equiv 0,\\
&\Big[\frac{\partial}{\partial\, \tau} - \sum_s \bar{n}_{(1)(s)}\, \frac{\partial}{\partial\, \sigma^s}\Big]\, \rho \equiv 0.
\end{aligned}
\tag{4.27}
$$

Let us remark that in the non relativistic limit $U^A \mapsto (1, U^r_{(nr)})$, $w_r \mapsto -U^r_{(nr)}$ and $\rho \mapsto \rho_{(nr)}$, where $\rho_{(nr)}$ is the non relativistic mass density. Therefore, in the non relativistic limit, the previous equations become the *non relativistic Euler's equations* (with $n_{(1)}$ being the Newtonian gravitational potential) and the *mass conservation equation*

$$
\begin{aligned}
&\frac{\partial\, U^r_{(nr)}}{\partial\, \tau} - \sum_s U^s_{(nr)}\, \frac{\partial\, U^r_{(nr)}}{\partial\, \sigma^s} = - \frac{\partial\, n_{(1)}}{\partial\, \sigma^r},\\
&\Big[\frac{\partial}{\partial\, \tau} + \sum_s U^s_{(nr)}\, \frac{\partial}{\partial\, \sigma^s}\Big]\, \rho_{(nr)} = 0.
\end{aligned}
\tag{4.28}
$$

V. KINEMATICAL AND DYNAMICAL ASPECTS OF THE DUST

In this Section we study the congruence of the dust flux lines, its connection with the skew congruence associated with the 3+1 splitting and the Eulerian point of view for the dust. Then we will define the acceleration, the expansion, the shear and the vorticity of the dust flux lines. Finally we will face the problem of how to select the subset of the irrotational motions of the dust and we will study the connection between the resulting dust 3-spaces and the 3-spaces of the Eulerian observers of the 3+1 splitting.

A. The Congruence of Dust Flux Lines

As said in Section IIC there are two congruences of time-like observers associated with each 3+1 splitting of space-time, i.e. with every global non-inertial frame. Their unit 4-velocities are $l^\mu(\tau,\vec{\sigma}) = \Big(z^\mu_A\, l^A\Big)(\tau,\vec{\sigma})$ and $v^\mu(\tau,\vec{\sigma}) = \frac{z^\mu_\tau}{\sqrt{\epsilon\, {}^4g_{\tau\tau}}}(\tau,\vec{\sigma}) = \Big(z^\mu_A\, v^A\Big)(\tau,\vec{\sigma})$.

When the dust (or every type of perfect fluid) is present we also have the congruence of the time-like flux lines with unit time-like 4-velocity $U^\mu(\tau,\vec{\sigma}) = z^\mu_A(\tau,\vec{\sigma})\, U^A(\tau,\vec{\sigma})$, which in general is not surface-forming.

The flux line of the dust emanating from $\vec{\sigma}_o$ at $\tau = 0$ are denoted

$$\zeta^\mu_{(0,\vec{\sigma}_o)}(\lambda) = z^\mu\Big(\tau(\lambda) = \zeta^\tau_{(0,\vec{\sigma}_o)}(\lambda);\, \Sigma^r(\tau(\lambda),\vec{\sigma}_o) = \zeta^r_{(0,\vec{\sigma}_o)}(\lambda)\Big). \tag{5.1}$$

Here λ is an affine parameter, $\zeta^A_{(0,\vec{\sigma}_o)}(\lambda) = \Big(\tau(\lambda); \vec{\Sigma}(\tau(\lambda),\vec{\sigma}_o)\Big)$, $\tau(0) = 0$, $\vec{\Sigma}(0,\vec{\sigma}_o) = \vec{\sigma}_o$, $\zeta^A_{(0,\vec{\sigma}_o)}(0) = (0,\vec{\sigma}_o)$. This equation defines *the 3-coordinates $\Sigma^r(\tau,\vec{\sigma}_o)$ identifying the location at time τ of the flux line emanating from σ^r_o at $\tau = 0$*.

The flux lines are the integral curves of the unit time-like 4-velocity, namely they are the solutions of the equations

$$\frac{d\,\zeta^A_{(0,\vec{\sigma}_o)}(\lambda)}{d\,\lambda} = U^A\Big(\tau = \tau(\lambda) = \zeta^\tau_{(0,\vec{\sigma}_o)}(\lambda);\, \sigma^r = \Sigma^r(\tau(\lambda),\vec{\sigma}_o) = \zeta^r_{(0,\vec{\sigma}_o)}(\lambda)\Big). \tag{5.2}$$

If we would impose the three gauge fixings on the gravitational field

$$\begin{aligned}
&U^r(\tau,\vec{\sigma}) \approx 0,\\
&\qquad \Downarrow \qquad (4.12),\\
&\bar{n}_{(a)}(\tau,\vec{\sigma}) \approx -\frac{(1+n)\,\tilde{\phi}^{-1/3}\, Q_a^{-1}\, \sum_{si} V_{sa}\,\partial_s\,\alpha^i\,\Pi_i}{\sqrt{\mu^2\,(J^\tau)^2 + \tilde{\phi}^{-2/3}\,\sum_{arsij}\, Q_a^{-2}\, V_{ra}\, V_{sa}\,\partial_r\,\alpha^i\,\partial_s\,\alpha^j\,\Pi_i\,\Pi_j}}(\tau,\vec{\sigma})),\\
&\alpha^i(\tau,\vec{\sigma}) \approx \alpha^i(\vec{\sigma}), \quad from \quad Eq.(4.15),
\end{aligned} \tag{5.3}$$

we would get $U^\mu(\tau,\vec{\sigma}) = v^\mu(\tau,\vec{\sigma})$, namely the fluid 4-velocity would coincide with the 4-velocity of the observers of the skew congruence of the 3+1 splitting, whose world-line passing through σ^r_o is now also a flux line with *constant* 3-coordinate σ^r_o

$$
\begin{aligned}
x^\mu_{\vec{\sigma}_o}(\tau(\lambda)) &= \zeta^\mu_{(0,\vec{\sigma}_o)}(\lambda) = z^\mu(\tau(\lambda),\vec{\sigma}_o),\\
&\Rightarrow \qquad \zeta^r_{(0,\vec{\sigma}_o)}(\lambda) = \Sigma^r(\tau(\lambda),\vec{\sigma}_o) = \sigma^r_o. \qquad (5.4)
\end{aligned}
$$

Since the conditions (5.3) determine the inertial shift functions, it means that there is a choice of 3-coordinates $\theta^i(\tau,\vec{\sigma})$ on the 3-spaces Σ_τ implying this coincidence [8].

For $\tau = \lambda$ Eqs. (5.3) are consistent with the first half of the Hamilton equations (4.14). In this case we have $\Sigma^r(\tau,\vec{\sigma}_o) = \sigma^r_o$ and $\alpha^i(\tau,\vec{\sigma}) = f^i(\vec{\sigma})$.

The comoving coordinates of the gauges (5.3) are usually used in *cosmology* when the matter is dust. See for instance Ref.[17] for the dust and Refs. [7, 18] for more general fluids.

B. Other Aspects of the Dust Flux Lines

As shown in Ref.[11], instead of the (comoving) Lagrangian coordinates $\alpha^i(\tau,\vec{\sigma})$ describing the fluid in the 3-spaces Σ_τ, we can use the set of Eulerian coordinates $\Sigma^r(\tau,\vec{\sigma}_o)$ appearing in the definition (5.1) of the fluid flux lines.

If we identify the affine parameter λ with the time τ of the observer origin of the radar 4-coordinates, i.e. if we put $\lambda = \tau$, the flux lines are described by the radar 4-coordinates $\zeta^A_{(0,\vec{\sigma}_o)}(\tau) = \Big(\tau;\, \vec{\Sigma}(\tau,\vec{\sigma}_o)\Big)$.

Since Eq.(4.15), i.e. $U^A(\tau,\vec{\sigma})\,\partial_A\,\alpha^i(\tau,\vec{\sigma}) = 0$, imply that the scalar fields $\alpha^i(\tau,\vec{\sigma})$ can be interpreted as *labels* assigned to the flux lines (they are constant along the flux lines), we can write

$$
\alpha^i\Big(\tau, \vec{\Sigma}(\tau,\vec{\sigma}_o)\Big) = \alpha^i(0,\vec{\sigma}_o) \overset{def}{=} \alpha^i_o(\vec{\sigma}_o). \qquad (5.5)
$$

The quantities $\alpha^i_o(\vec{\sigma}_o)$ are Cauchy data on the Cauchy surface $\Sigma_{\tau=0}$ at $\tau = 0$, whose 3-coordinates are denoted $\sigma^r_o = \Sigma^r(0,\vec{\sigma}_o)$. If we invert the relation $\sigma^r = \Sigma^r(\tau,\vec{\sigma}_o)$ to get $\sigma^r_o = g^r_{\vec{\Sigma}}(\tau,\sigma)$, we have

$$
\alpha^i(\tau,\vec{\sigma}) = \alpha^i(0,\, \vec{g}_{\vec{\Sigma}}(\tau,\vec{\sigma})) = \alpha^i_o(\vec{g}_{\vec{\Sigma}}(\tau,\vec{\sigma})). \qquad (5.6)
$$

Since the conserved particle number is $\mathcal{N} = \int_{V_{(\alpha)}(\tau)} d^3\sigma\, J^\tau(\alpha^i(\tau,\vec{\sigma}))$, see after Eq.(3.2), on $\Sigma_{\tau=0}$ the initial number density is

$$
\tilde{n}_o(\vec{\sigma}_o) = J^\tau(\alpha^i(0,\vec{\sigma})) = det\,\Big(\frac{\partial\,\alpha^i_o(\vec{\sigma}_o)}{\partial\,\sigma^r_o}\Big). \qquad (5.7)
$$

If the fluid has compact support we can take $\alpha^i_o(\vec{\sigma}_o) = \sigma^{r=i}_o$, so that $\tilde{n}_o(\vec{\sigma}_o) = 1$.

[8] As said in Appendix A, the τ-preservation of the three gauge fixing for the angles θ^i implies three equations for the determination of the shift functions.

C. The Eulerian Canonical Coordinates

The flux lines of Eq.(5.1), $\zeta^A_{(0,\vec{\sigma}_o)}(\tau) = \Big(\tau,\, \vec{\Sigma}(\tau, \vec{\sigma}_o)\Big)$ with $\lambda = \tau$, are identified by the 3-coordinates $\sigma^r = \Sigma^r(\tau, \vec{\sigma}_o)$, where σ^r_o are the 3-coordinates on the Cauchy surface $\Sigma_{\tau=0}$.

As shown in Ref.[11], there is a point canonical transformation allowing to pass from the (comoving) Lagrangian canonical coordinates $\alpha^i(\tau, \vec{\sigma})$, $\Pi_i(\tau, \vec{\sigma})$, to a set of Eulerian canonical coordinates $\Sigma^r(\tau, \vec{\sigma}_o)$, $K^r(\tau, \vec{\sigma}_o)$. Therefore the new fluid coordinates coincide with the 3-coordinates σ^r_o (and not with $\alpha^i_o(\vec{\sigma}_o)$) on the Cauchy surface $\Sigma_{\tau=0}$ and for $\tau > 0$ they label the flux lines with $\Sigma^r(\tau, \vec{\sigma}_o)$ instead that with $\alpha^i(\tau, \vec{\sigma}) = \alpha^i_o(\vec{g}_{\vec{\Sigma}}(\tau, \vec{\sigma}_o))$. The initial number density $\tilde{n}_o(\vec{\sigma}_o)$ is assumed to be a known function given as part of the Cauchy data. Also $\alpha^i_o(\vec{\sigma}_o)$ is known from the Cauchy data.

The relation between the old coordinates and the new ones is obtained by rewriting the relation $\alpha^i(\tau, \vec{\sigma}) = \alpha^i_o(\vec{g}_{\vec{\Sigma}}(\tau, \vec{\sigma}_o))$ in the following form

$$\alpha^i(\tau, \vec{\sigma}) = \int d^3\sigma_o \, \det\left(\frac{\partial \Sigma^u}{\partial \sigma^v_o}\right)(\tau, \vec{\sigma}_o)\, \delta^3(\sigma^r - \Sigma^r(\tau, \vec{\sigma}_o))\, \alpha^i_o(\vec{\sigma}_o). \tag{5.8}$$

By using $\alpha^i(\tau, \vec{\Sigma}(\tau, \vec{\sigma}_o)) = \alpha^i_o(\vec{\sigma}_o)$, it can be shown that the inverse of Eq.(5.8) is

$$\Sigma^r(\tau, \vec{\sigma}_o) = \int d^3\sigma \, \det\left(\frac{\partial\, \alpha^i}{\partial\, \sigma^u}\right)(\tau, \vec{\sigma})\, \delta^3(\alpha^i_o(\vec{\sigma}_o) - \alpha^i(\tau, \vec{\sigma}))\, \sigma^r. \tag{5.9}$$

The generating functional and the new momenta of the point canonical transformation are

$$\begin{aligned}
\Phi\Big[\Sigma^r, \Pi_i\Big] &= \int d^3\sigma\, \Pi_i(\tau, \vec{\sigma}) \int d^3\sigma_o \, \det\left(\frac{\partial \Sigma^u}{\partial \sigma^v_o}\right)(\tau, \vec{\sigma}_o)\, \delta^3(\sigma^r - \Sigma^r(\tau, \vec{\sigma}_o))\, \alpha^i_o(\vec{\sigma}_o),\\
&\Downarrow\\
K_r(\tau, \vec{\sigma}_o) &= \frac{\delta \Phi}{\delta \Sigma^r(\tau, \vec{\sigma}_o)} =\\
&= -\det\left(\frac{\partial \Sigma^u}{\partial \sigma^v_o}\right)(\tau, \vec{\sigma}_o)\, \frac{\partial \sigma^u_o}{\partial \Sigma^r}(\tau, \vec{\sigma}_o)\, \frac{\partial \alpha^i_o(\vec{\sigma}_o)}{\partial \sigma^u_o} \int d^3\sigma\, \Pi_i(\tau, \vec{\sigma})\, \delta^3(\sigma^s - \Sigma^s(\tau, \vec{\sigma}_o)).
\end{aligned} \tag{5.10}$$

Instead for the inverse canonical transformation we have the following generating functional

$$\begin{aligned}
\Phi'\Big[\alpha^i, K_r\Big] &= \int d^3\sigma_o\, K_r(\tau,\vec{\sigma}) \int d^3\sigma\, \det\left(\frac{\partial\alpha^i}{\partial\sigma^u}\right)(\tau,\vec{\sigma})\, \delta^3(\alpha^i_o(\vec{\sigma}_o) - \alpha^i(\tau,\vec{\sigma}))\, \sigma^r,\\
&\Downarrow\\
\Pi_i(\tau,\vec{\sigma}) &= \frac{\delta\Phi'}{\delta\alpha^i(\tau,\vec{\sigma})} =\\
&= -\det\left(\frac{\partial\alpha}{\partial\sigma}\right)(\tau,\vec{\sigma})\, \frac{\partial\sigma^r}{\partial\alpha^i}(\tau,\vec{\sigma}) \int d^3\sigma_o\, K_r(\tau,\vec{\sigma}_o)\, \delta^3(\alpha^i_o(\sigma_o) - \alpha(\tau,\vec{\sigma})).
\end{aligned}\tag{5.11}$$

It can be checked that these expressions for the momenta are one the inverse of the other.

By using the notations introduced after Eq.(3.6), i.e. $\mathcal{I}_{ir}(\tau,\vec{\sigma}) = \partial_r\,\alpha^i(\tau,\vec{\sigma})$, we get $\sum_r \mathcal{I}_{ir}(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))\, \frac{\partial\Sigma^r(\tau,\vec{\sigma}_o)}{\partial\sigma^s_o} = \frac{\partial\alpha^i_o(\vec{\sigma}_o)}{\partial\sigma^s_o}$ with $\alpha^i_o(\vec{\sigma}_o)$ defined in Eq.(5.5). Since we have $0 = \frac{d\,\alpha^i_o(\vec{\sigma}_o)}{d\tau} = \frac{d\alpha^i(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))}{d\tau} = \frac{\partial\,\alpha^i(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))}{\partial\,\tau} + \sum_r \mathcal{I}_{ir}(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))\, \frac{\partial\,\Sigma^r(\tau,\vec{\sigma}_o)}{\partial\,\tau}$, we get from Eq.(3.2) $(\partial_\tau\,\alpha^i = -\sum_r \frac{J^r}{J^\tau}\,\mathcal{I}_{ir})$ the following results

$$\begin{aligned}
\frac{\partial\,\Sigma^r(\tau,\vec{\sigma}_o)}{\partial\,\tau} &= -\sum_i \mathcal{I}^{-1}_{ir}(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))\, \frac{\partial\,\alpha^i(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))}{\partial\,\tau} = \frac{J^r(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))}{J^\tau(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))},\\
J^\tau(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o)) &= -\det(\partial_r\,\alpha^i)(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o)) = -\det{}^{-1}\left(\frac{\partial\,\Sigma^u}{\partial\,\sigma^v_o}\right)(\tau,\vec{\sigma}_o)\,\det\left(\frac{\partial\,\alpha^i_o(\vec{\sigma}_o)}{\partial\,\sigma^s_o}\right) =\\
&= \tilde{n}_o(\vec{\sigma}_o)\,\det{}^{-1}\left(\frac{\partial\,\Sigma^u}{\partial\,\sigma^v_o}\right)(\tau,\vec{\sigma}_o), \quad\Rightarrow\quad J^\tau(\tau,\vec{\sigma}) = \Big(\tilde{n}(\vec{\sigma}_o)\,\det{}^{-1}\left(\frac{\partial\,\Sigma^r}{\partial\,\sigma^s_o}\right)\Big)\Big|_{\vec{\sigma}_o=\vec{g}_\Sigma(\tau,\vec{\sigma})},\\
J^r(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o)) &= \tilde{n}_o(\vec{\sigma}_o)\,\det{}^{-1}\left(\frac{\partial\,\Sigma^u}{\partial\,\sigma^v_o}\right)(\tau,\vec{\sigma}_o)\,\frac{\partial\,\Sigma^r(\tau,\vec{\sigma}_o)}{\partial\,\tau}.
\end{aligned}\tag{5.12}$$

As shown in Ref.[11], these results imply the following form of the mass density of the dust

$$\begin{aligned}
\mathcal{M}(\tau,\vec{\sigma}) &= \Big[\det{}^{-1}\left(\frac{\partial\,\Sigma^r(\tau,\vec{\sigma}_o)}{\partial\,\sigma^s_o}\right)\sqrt{\mu^2\,\tilde{n}^2_o(\vec{\sigma}_o) + {}^3g^{rs}(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))\,K_r(\tau,\vec{\sigma}_o)\,K_s(\tau,\vec{\sigma}_o)}\Big]\Big|_{\vec{\sigma}_o=\vec{g}_\Sigma(\tau,\vec{\sigma})} =\\
&= \int d^3\sigma_o\,\delta^3(\sigma^r - \Sigma^r(\tau,\vec{\sigma}_o))\sqrt{\mu^2\,\tilde{n}^2_o(\vec{\sigma}_o) + {}^3g^{rs}(\tau,\vec{\Sigma}(\tau,\vec{\sigma}_o))\,K_r(\tau,\vec{\sigma}_o)\,K_s(\tau,\vec{\sigma}_o)}.
\end{aligned}\tag{5.13}$$

Instead for the mass current density we get the following result valid for any kind of perfect fluid (for the dust we have $\mathcal{M}_r = -\mu\,J^\tau\,U_r$)

$$\mathcal{M}_r(\tau,\vec\sigma) = \sum_i \partial_r\,\alpha^i(\tau,\vec\sigma)\,\Pi_i(\tau,\vec\sigma) = -\Big[\det{}^{-1}\Big(\frac{\partial\,\Sigma^u(\tau,\vec\sigma_o}{\partial\,\sigma^v_o)}\Big)\;K_r(\tau,\vec\sigma_o)\Big]\Big|_{\vec\sigma_o=\vec g_\Sigma(\tau,\vec\sigma)},$$

$$\Rightarrow\qquad U_r(\tau,\vec\sigma) = \frac{K_r(\tau,\vec g_{\vec\Sigma}(\tau,\vec\sigma)}{\mu\,\tilde n_o(\vec g_{\vec\Sigma}(\tau,\vec\sigma))}. \tag{5.14}$$

D. Acceleration, Expansion, Shear and Vorticity of the Dust

The covariant derivative of the dust 4-velocity $U^A(\tau,\vec\sigma)$, whose Hamiltonian expression is given in Eqs.(4.12), allows to find the associated acceleration $a^A_{(U)}$, expansion $\theta_{(U)}$, shear $\sigma_{(U)AB}$ and vorticity $\omega_{(U)AB}$ of the dust. The analogous quantities for the Eulerian observers, associated with the 3+1 splitting and having the unit normal $l^A(\tau,\vec\sigma)$ to the 3-spaces Σ_τ, are given in Eqs.(A5)-(A10) of Appendix A.

For the dust we get

$$\begin{aligned}
{}^4\nabla_A\,\epsilon\,U_B &= \epsilon\,U_A\,a_{(U)B} + \sigma_{(U)AB} + \frac{1}{3}\,\theta_{(U)}\,{}^3h_{(U)AB} - \omega_{(U)AB},\qquad {}^3h_{(U)AB} = {}^4g_{AB} - \epsilon\,U_A\,U_B,\\
a^A_{(U)} &= U^B\,{}^4\nabla_B\,U^A \circeq 0,\qquad a_{(U)A} = {}^4g_{AB}\,a^A_{(U)},\qquad a^A_{(U)}\,U_A = 0,\\
\theta_{(U)} &= {}^4\nabla_A\,U^A,\\
\sigma_{(U)AB} &= \sigma_{(U)BA} = -\frac{\epsilon}{2}\,(a_{(U)A}\,U_B + a_{(U)B}\,U_A) + \frac{\epsilon}{2}\,({}^4\nabla_A\,U_B + {}^4\nabla_B\,U_A) - \frac{1}{3}\,\theta_{(U)}\,{}^3h_{(U)AB} \circeq,\\
&\circeq \frac{\epsilon}{2}\,({}^4\nabla_A\,U_B + {}^4\nabla_B\,U_A) - \frac{1}{3}\,\theta_{(U)}\,{}^3h_{(U)AB},\qquad \sigma_{(U)AB}\,U^B = 0,\\
\omega_{(U)AB} &= -\omega_{(U)BA} = \eta_{ABCD}\,\omega^C_{(U)}\,U^D =\\
&= -\frac{\epsilon}{2}\,(a_{(U)A}\,U_B - a_{(U)B}\,U_A) - \frac{\epsilon}{2}\,({}^4\nabla_A\,U_B - {}^4\nabla_B\,U_A) \circeq -\frac{\epsilon}{2}\,({}^4\nabla_A\,U_B - {}^4\nabla_B\,U_A),\\
&\qquad \omega^A_{(U)} = \frac{1}{2}\,\eta^{ABCD}\,\omega_{(U)BC}\,U_D,\qquad \omega_{(U)AB}\,U^B = 0,\qquad \omega^A_{(U)}\,U_A = 0.
\end{aligned} \tag{5.15}$$

We have explicitly shown the consequences of the fact that the equations of motion (4.17) imply the vanishing of the acceleration $a^A_{(U)}(\tau,\vec\sigma)\circeq 0$.

In many contexts, especially in cosmology, one considers irrotational dust, having $\omega_{(U)AB}(\tau,\vec\sigma) = 0$. In this case the unit time-like 4-velocity $U^A(\tau,\vec\sigma)$ is surface forming, namely there is a 3+1 splitting of space-time whose 3-spaces $\Sigma_{(U)\tau}$ are orthogonal to $U^A(\tau,\vec\sigma)$. In these cases there should exist gauges of canonical gravity in which the congruence of Eulerian observers coincides with the dust congruence of observers.

However our dynamical dust has in general non-vanishing vorticity. If we try to impose the condition $\omega_{(U)AB}(\tau,\vec{\sigma})=0$ by asking to have $U_A(\tau,\vec{\sigma})\approx l_A(\tau,\vec{\sigma})=\epsilon\,\Big(1+n(\tau,\vec{\sigma})\Big)\,(1;0)$, we get the conditions $U_r(\tau,\vec{\sigma})\approx 0$. But from Eqs.(4.13) this implies $\sum_i\,\partial_r\,\alpha^i(\tau,\vec{\sigma})\,\Pi_i(\tau,\vec{\sigma})\approx 0$, i.e. $\Pi_i(\tau,\vec{\sigma})\approx 0$ since $J^\tau(\tau,\vec{\sigma})\neq 0$. Then the second of Eqs.(4.14) gives $\sum_r\,\partial_r\,\Big((1+n)\,\epsilon^{ruv}\,\epsilon_{ijk}\,\partial_u\,\alpha^j\,\partial_v\,\alpha^k\Big)(\tau,\vec{\sigma})\approx 0$, i.e. three conditions on the lapse function implying $n=n(\tau)$.

To understand better this restriction, in the next Subsection we will explore how to implement the condition $\omega_{(U)AB}(\tau,\vec{\sigma})=0$ as a restriction on the space of solution of the dust motion. Then, after this identification, we will explore which gauge-fixing on canonical gravity will imply $U^A\approx l^A$ for the irrotational motions of the dust.

E. The Subset of Irrotational Dust Motions

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From Eqs.(5.15) we get that the conditions $\omega_{(U)AB}(\tau,\vec{\sigma})\approx 0$ are implied by $\omega_{(U)rs}(\tau,\vec{\sigma})\approx 0$, because we have $\omega_{(U)\tau r}(\tau,\vec{\sigma})=\sum_s\,\Big(\omega_{(U)rs}\,\frac{U^s}{U^\tau}\Big)(\tau,\vec{\sigma})\approx 0$.

Therefore we must study $\omega_{(U)rs}\,{\buildrel \circ \over =}\,-\frac{\epsilon}{2}\,({}^4\nabla_r\,U_s-{}^4\nabla_s\,U_r)=-\frac{\epsilon}{2}\,(\partial_r\,U_s-\partial_s\,U_r)\,{\buildrel {def} \over =}\,-\frac{\epsilon}{2}\,\Omega_{(U)rs}\approx 0$ as a restriction on the solutions for the dust motion.

Let us remark that this restriction is preserved in time by the Hamilton equations, because Eqs.(4.13), (4.14) and (4.12) imply $\frac{\partial\,\Omega_{(U)rs}(\tau,\vec{\sigma})}{\partial\,\tau}\,{\buildrel \circ \over =}\,\{\Omega_{(U)rs}(\tau,\vec{\sigma}),H_D\}=\Big[-\frac{\partial}{\partial\,\sigma^r}\,\Big(\Omega_{(U)su}\,\frac{U^u}{U^\tau}\Big)+\frac{\partial}{\partial\,\sigma^s}\,\Big(\Omega_{(U)ru}\,\frac{U^u}{U^\tau}\Big)\Big]\,(\tau,\vec{\sigma})\approx 0$. Therefore it is enough to impose the condition $\Omega_{(U)rs}(0,\vec{\sigma})\approx 0$ on the Cauchy surface $\Sigma_{\tau=0}$. This also shows that these conditions can be interpreted as first-class constraints to be added by hand to select the family of irrotational motions of the fluid.

Let us remark that $\Omega_{(U)rs}\approx 0$ are only two independent conditions, because we have the identity $\partial_r\,\Omega_{(U)uv}+\partial_u\,\Omega_{(U)vr}+\partial_v\,\Omega_{(U)ru}=0$.

If we make the following non-covariant decomposition of $U_r(\tau,\vec{\sigma})$ [9] ($\triangle=\sum_r\,\partial_r^2$; the function $c(\tau)$ is required by the boundary conditions at spatial infinity and will make the gauge fixing later introduced in Eq.(5.25) well defined)

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[9] Compare with the analogous decomposition of the electro-magnetic vector potential in Refs. [10], [3], where it is shown that one must firstly fix the 3-coordinates on the 3-space Σ_τ and then make the non-covariant decomposition.

$$U_r(\tau,\vec{\sigma}) \,=\, \partial_r\, S(\tau,\vec{\sigma}) + U_{\perp\, r}(\tau,\vec{\sigma}),$$

$$S(\tau,\vec{\sigma}) = c(\tau) + \sum_r \frac{\partial_r}{\triangle}\, U_r(\tau,\vec{\sigma}) = c(\tau) - \frac{1}{\mu} \sum_{ri} \frac{\partial_r}{\triangle}\, \frac{\partial_r\, \alpha^i\, \Pi_i}{J^\tau}(\tau,\vec{\sigma}),$$

$$U_{\perp\, r}(\tau,\vec{\sigma}) = -\sum_v \frac{\partial_v}{\triangle}\, \Omega_{(U)vr}(\tau,\vec{\sigma}), \qquad \sum_r \partial_r\, U_{\perp\, r}(\tau,\vec{\sigma}) = 0, \tag{5.16}$$

the two conditions of vanishing vorticity are

$$U_{\perp\, r}(\tau,\vec{\sigma}) \approx 0, \qquad \Rightarrow \quad U_r(\tau,\vec{\sigma}) \approx \partial_r\, S(\tau,\vec{\sigma}). \tag{5.17}$$

These are the two first-class constraints to be added by hand to eliminate the states of motion of the dust with non-zero vorticity. They show that two of the three pairs of canonical variables $\alpha^i(\tau,\vec{\sigma})$, $\Pi_i(\tau,\vec{\sigma})$, describing the dust are associated with vorticity. The addition of two suitable gauge fixing constraints (whose form is not known), so to get two pairs of second class constraints, would allow to go to Dirac brackets and to describe the irrotational dust with only one pair of canonical variables.

Since $S(\tau,\sigma)$ and $U_{\perp\, r}(\tau,\vec{\sigma})$ have non-trivial Poisson brackets (i.e. $\{S(\tau,\vec{\sigma}), S(\tau,\vec{\sigma}\,')\} \not= 0$, $\{\Omega_{(U)rs}(\tau,\vec{\sigma}), \Omega_{(U)uv}(\tau,\vec{\sigma}\,')\} \not= 0$, $\{S(\tau,\vec{\sigma}), \Omega_{(U)uv}(\tau,\vec{\sigma}\,')\} \not= 0$), it is not easy to identify the two gauge fixings to be added. This is not easy also in the Eulerian point of view, where Eqs.(5.14) and (5.12) imply $U_r(\tau,\vec{\sigma}) = \frac{K_r(\tau,\vec{g}_{\vec{\Sigma}}(\tau,\vec{\sigma}))}{\mu\, \tilde{n}_o(\vec{g}_{\vec{\Sigma}}(\tau,\vec{\sigma}))}$. The presence of the initial number density and of the function $\vec{g}_{\vec{\Sigma}}(\tau,\vec{\sigma})$ imply a complicated expression for $U_{\perp\, r}(\tau,\vec{\sigma})$, so that the two gauge fixings will be complicated functions of $\Sigma^r(\tau,\vec{\sigma}_o)$.

However it is possible to identify the reduced phase space of the irrotational dust also without knowing the gauge fixings explicitly. Eqs.(4.12) imply

$$\{U_r(\tau,\vec{\sigma}), U_s(\tau,\vec{\sigma}\,')\} = -\Big(\frac{\Omega_{(U)rs}}{\mu\, \det(\partial_u\, \alpha^j)}\Big)(\tau,\vec{\sigma})\, \delta^3(\sigma^v - \sigma'^{\,v}),$$

$$\{\det(\partial_u\, \alpha^i(\tau,\vec{\sigma})), S(\tau,\vec{\sigma}\,')\} = -\frac{1}{\mu}\, \delta^3(\sigma^v - \sigma'^{\,v}),$$

$$\{\det(\partial_r\, \alpha^i(\tau,\vec{\sigma})), \Omega_{(U)uv}(\tau,\vec{\sigma}\,')\} = 0. \tag{5.18}$$

As a consequence we get

$$\{U_r(\tau,\vec{\sigma}), U_s(\tau,\vec{\sigma}\,')\}|_{\Omega_{(U)rs}=0} = 0,$$

$$\{S(\tau,\vec{\sigma}), S(\tau,\vec{\sigma}\,')\}|_{\Omega_{(U)rs}=0} = 0,$$

$$\{S(\tau,\vec{\sigma}), \Omega_{(U)uv}(\tau,\vec{\sigma}\,')\}\Big|_{\Omega_{(U)rs}=0} = 0. \tag{5.19}$$

Even if we do not know the expression of the two gauge fixings, it turns out that the reduced phase space of the irrotational dust is spanned by the two conjugate canonical variables

$$\nu(\tau,\vec{\sigma}) = \mu\, J^{\tau}(\tau,\vec{\sigma}) = \mu\, \det(\partial_r\, \alpha^i(\tau,\vec{\sigma})), \qquad S(\tau,\vec{\sigma}) = \sum_r \partial_r\, U_r(\tau,\vec{\sigma}), \tag{5.20}$$

because, due to Eqs.(5.18) and (5.19), their Dirac brackets are

$$\begin{aligned}
&\{\nu(\tau,\vec{\sigma}), \nu(\tau,\vec{\sigma}')\}^* = \{\nu(\tau,\vec{\sigma}), \nu(\tau,\vec{\sigma}')\}_{\Omega_{(U)rs}=0} = 0,\\
&\{S(\tau,\vec{\sigma}), S(\tau,\vec{\sigma}')\}^* = \{S(\tau,\vec{\sigma}), S(\tau,\vec{\sigma}')\}_{\Omega_{(U)rs}=0} = 0,\\
&\{\nu(\tau,\vec{\sigma}), S(\tau,\vec{\sigma}')\}^* = \{\nu(\tau,\vec{\sigma}), S(\tau,\vec{\sigma}')\}_{\Omega_{(U)rs}=0} = -\delta^3(\sigma^r - {\sigma'}^r).
\end{aligned}\tag{5.21}$$

While $\nu(\tau,\vec{\sigma})$ describes the numerical density of the dust ($\mathcal{N} = \frac{1}{\mu}\int d^3\sigma\, \nu(\tau,\vec{\sigma})$), the function $S(\tau,\vec{\sigma})$ allows to put the dust unit 4-velocity in the form (${}^3g^{rs} = \tilde{\phi}^{-2/3}\sum_a Q_a^{-2}\, V_{ra}\, V_{sa}$, $n^r = \sum_a \bar{n}_{(a)}\, Q_a^{-1}\, V_{ra}$)

$$\begin{aligned}
U_{\tau}(\tau,\vec{\sigma}) &= (1 + n(\tau,\vec{\sigma}))\, \sqrt{1 + {}^3g^{rs}(\tau,\vec{\sigma})\, \partial_r\, S(\tau,\vec{\sigma})\, \partial_s\, S(\tau,\vec{\sigma})} + n^r(\tau,\vec{\sigma})\, \partial_r\, S(\tau,\vec{\sigma}),\\
U_r(\tau,\vec{\sigma}) &= \partial_r\, S(\tau,\vec{\sigma}), \qquad U_A\, {}^4g^{AB}\, U_B = 1.
\end{aligned}\tag{5.22}$$

With the substitutions $J^{\tau} = \det(\partial_r\, \alpha^i) \mapsto \frac{1}{\mu}\, \nu$, $\partial_r \alpha^i\, \Pi_i \mapsto -\nu\, \partial_r\, S$, the part $\int d^3\sigma \Big((1 + n)\, \mathcal{M} + n^r\, \mathcal{M}_r\Big)(\tau,\vec{\sigma})$ of the Dirac Hamiltonian (4.13) takes the form

$$H^{(R)}_{dust} = \int d^3\sigma\, \nu(\tau,\vec{\sigma})\, \Big[(1 + n(\tau,\vec{\sigma}))\, \sqrt{1 + {}^3g^{rs}(\tau,\vec{\sigma})\, \partial_r\, S(\tau,\vec{\sigma})\, \partial_s\, S(\tau,\vec{\sigma})} + n^r(\tau,\vec{\sigma})\, \partial_r\, S(\tau,\vec{\sigma})\Big]. \tag{5.23}$$

This reduced Hamiltonian gives the following Hamilton equations in the reduced phase space of the irrotational dust

$$\begin{aligned}
\partial_{\tau}\, S(\tau,\vec{\sigma}) &\circeq \{S(\tau,\vec{\sigma}), H^{(R)}_{dust}\}^* = \Big((1 + n)\, \sqrt{1 + {}^3g^{rs}\, \partial_r\, S\, \partial_s\, S} + n^r\, \partial_r\, S\Big)(\tau,\vec{\sigma}) = U_{\tau}(\tau,\vec{\sigma}),\\
\partial_{\tau}\, \nu(\tau,\vec{\sigma}) &\circeq \{\nu, H^{(R)}_{dust}\}^* = -\Big(\partial_r\, \Big[\nu\, \Big((1 + n)\, \frac{{}^3g^{rs}\, \partial_s\, S}{\sqrt{1 + {}^3g^{rs}\, \partial_r\, S\, \partial_s\, S}} + n^r\Big)\Big]\Big)(\tau,\vec{\sigma}),
\end{aligned}\tag{5.24}$$

which allow to write $U_A(\tau,\vec{\sigma}) \circeq \partial_A\, S(\tau,\vec{\sigma})$ consistently with Eqs.(5.22). The second half of Eqs.(5.24) allow to check the constancy of the particle number: $\partial_{\tau}\, \mathcal{N} = \frac{1}{\mu}\, \int d^3\sigma\, \partial_{\tau}\, \nu(\tau,\vec{\sigma}) \circeq 0$.

The result $U_A(\tau, \vec{\sigma}) \stackrel{\circ}{=} \partial_A S(\tau, \vec{\sigma})$ shows that the unit time-like 4-velocity of the irrotational dust is orthogonal to the 3-spaces $S(\tau, \vec{\sigma}) = const$. Moreover one can check the validity of Eq.(4.17) in the reduced phase space, namely that the flux lines are geodesics.

If we add the constraint

$$\chi(\tau, \vec{\sigma}) = S(\tau, \vec{\sigma}) - \tilde{S}(\tau) \approx 0, \quad i.e. \quad c(\tau) \approx \tilde{S}(\tau), \tag{5.25}$$

implying $U_r(\tau, \vec{\sigma}) \approx 0$, $U_A(\tau, \vec{\sigma}) \approx l_A(\tau, \vec{\sigma})$ and $\Pi_i(\tau, \vec{\sigma}) \approx 0$ for the irrotational dust in the original phase space, its preservation in time produces the constraint

$$\begin{aligned}
\partial_\tau \chi(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \{\chi(\tau, \vec{\sigma}), H^{(R)}_{dust}\}^* - \frac{d\,\tilde{S}(\tau)}{d\tau} = U_\tau(\tau, \vec{\sigma}) - \frac{d\,S(\tau)}{d\tau} \approx \\
&\approx 1 + n(\tau, \vec{\sigma}) - \frac{d\,\tilde{S}(\tau)}{d\tau} \approx 0, \\
&\Downarrow \qquad (5.24) \\
&\partial_\tau \nu(\tau, \vec{\sigma}) \approx -\sum_r \partial_r \Big(\nu\, n^r\Big)(\tau, \vec{\sigma}), \qquad n = n(\tau).
\end{aligned} \tag{5.26}$$

But this means that there is a gauge fixing for the inertial gauge variable ${}^3K(\tau, \vec{\sigma})$ of canonical gravity whose τ-preservation implies that the lapse is only a function of time. As a consequence there is a 3+1 splitting of space-time whose 3-spaces coincide with the ones of the irrotational dust.

If before the restriction to irrotational dust we had chosen the 3-coordinates $\theta^i(\tau, \vec{\sigma})$ such that Eqs.(5.3) hold, i.e. $U^r(\tau, \vec{\sigma}) \approx 0$, then Eqs.(5.26) imply $\bar{n}_{(a)}(\tau, \vec{\sigma}) \approx 0$ and $\partial_\tau \nu(\tau, \vec{\sigma}) \approx 0$.

This, with the extra condition $n(\tau) = 0$, is often the starting point in cosmology with irrotational dust and comoving 3-coordinates [7, 17]. In this formulation the irrotational dust is not described by canonical coordinates like $\alpha^i(\tau, \vec{\sigma})$, $\Pi_i(\tau, \vec{\sigma})$, but only by a function $\rho^{'}(\tau, \vec{\sigma})$ such that the energy-momentum tensor is $T^{\mu\nu} = \rho^{'}\, U^\mu\, U^\nu$. This function satisfies the Bianchi identities $T^{\mu\nu}{}_{;\nu} = 0$, which imply [10] $\partial_\tau\, [\rho^{'}\, \tilde{\phi}] = 0$ ($\tilde{\phi} = \sqrt{det\, {}^3g}$) and the result $\partial_\tau\, \rho^{'} = {}^3K\, \rho^{'}$.

In our approach we have the function $\bar{\rho}$ of Eqs.(4.12), whose Hamilton equations are induced by Eqs.(4.14) and by the Hamilton equations for the gravitational field [2]. By using the Hamiltonian Bianchi identities of Ref.[2] and by restricting to irrotational dust satisfying both $U_r \approx 0$ and $U^r \approx 0$ (i.e. with $n \approx \bar{n}_{(a)} \approx 0$) we recover $\partial_\tau\, \bar{\rho} = {}^3K\, \bar{\rho}$.

Finally let us remark that by using Eqs.(5.15) we could also study: i) dust without expansion: it requires the study of the equation ${}^4\nabla\, U_A \approx 0$; ii) shear-free dust: one should study the equations $\frac{\epsilon}{2}\, ({}^4\nabla_A\, U_B + {}^4\nabla_B\, U_A) \approx \frac{1}{3}\, \theta_{(U)}\, {}^3h_{(U)AB}$.

[10] This has the consequence that the mass inside an averaging volume is constant $M_{\mathcal{V}}(\tau) = \int_{\mathcal{V}} d^3\sigma [\rho\, \tilde{\phi}](\tau, \sigma) = M_{\mathcal{V}}(\tau_o)$.

F. About Irrotational Motions in Arbitrary Perfect Fluids

Let us make some remarks on the existence of irrotational motions for arbitrary perfect fluids like the one discussed in Appendix C.

In Ref. [19] it is shown that Eqs.(5.15) together with the definition $\Big({}^4\nabla_A\, {}^4\nabla_B - {}^4\nabla_B\, {}^4\nabla_A\Big)\, U_C = -{}^4R_{CDAB}\, U^D$ of the 4-Riemann tensor imply the following equations for acceleration, expansion, shear and vorticity ($[AB]$ means anti-symmetrization)

$$
\begin{aligned}
&{}^3h_{(U)A}{}^{C}\, {}^3h_{(U)B}{}^{D}\, \Big(U^E\, \nabla_E\, \omega_{(U)CD} - {}^4\nabla_{\big[C}\, a_{(U)D\big]}\Big) + \\
&\qquad + 2\, \sigma_{(U)E\big[A}\, \omega^E_{(U)B\big]} + \frac{2}{3}\, \theta_{(U)}\, \omega_{(U)AB} = 0, \\
&{}^4\nabla_{\big[C}\, \omega_{(U)AB\big]} + {}^4\nabla_{\big[C}\, a_{(U)A}\, U_{B\big]} + a_{(U)\big[A}\, \omega_{(U)BC\big]} = 0.
\end{aligned}
\tag{5.27}
$$

Let us look whether a perfect fluid with arbitrary equation of state may admit irrotational motions, namely motions with null vorticity

$$
\omega_{(U)AB} \equiv 0. \tag{5.28}
$$

While the second of Eqs.(5.27) is identically satisfied, the first of Eqs.(5.27) implies that *the neccessary condition for the existence of irrotational motions* is

$$
{}^3h_{(U)A}{}^{C}\, {}^3h_{(U)B}{}^{C}\, {}^4\nabla_{\big[C}\, a_{(U)D\big]} \equiv 0. \tag{5.29}
$$

Therefore Eq.(5.29) is a restriction on the equation of state of the fluid, giving the pressure as a function of the energy density. Moreover Eq.(5.29) must be compatible with the relativistic Euler equations (see Eqs. (3.16) and (3.17)).

This problem was considered in Ref.[18], where it was shown that if for a family of motions the 4-velocity of the fluid admits the following parametrization in terms of a scalar function $S(\tau, \vec{\sigma})$

$$
\begin{aligned}
U_A(\tau, \vec{\sigma}) \;&=\; \frac{1}{h(\tau, \vec{\sigma})}\, \nabla_A\, S(\tau, \vec{\sigma}), \\
&\qquad h(\tau, \vec{\sigma}) = \sqrt{g^{AB}(\tau, \vec{\sigma})\, \nabla_A\, S(\tau, \vec{\sigma})\, \nabla_B\, S(\tau, \vec{\sigma})},
\end{aligned}
\tag{5.30}
$$

then the acceleration turns out to depend only on the normalization function $h(\tau, \vec{\sigma})$

$$
\begin{aligned}
a_{(U)B}(\tau, \vec{\sigma}) \;&=\; U^A(\tau, \vec{\sigma})\, \nabla_A\, U_B(\tau, \vec{\sigma}) = \\
&=\; \Big(\delta_B{}^A - U_B(\tau, \vec{\sigma})\, U^A(\tau, \vec{\sigma})\Big)\, \nabla_A\, \ln\, h(\tau, \vec{\sigma}),
\end{aligned}
\tag{5.31}
$$

and the condition (5.29) is satisfied. For the dust Eqs.(5.22) and (5.24) show that Eq.(5.30) is satisfied with $h(\tau, \vec{\sigma}) = 1$.

As shown in Ref.[18] barotropic fluids, with equation of state $\rho = \rho(p)$, admit a family of irrotational motions for which Eqs.(5.30) and (5.31), and then (5.29), are valid (the function h being proportional to $\Pi(p) = \int \frac{dp}{p+\rho(p)}$).

VI. CONCLUSION

Brown's formulation [9] of perfect fluids allows to describe them only in terms of three Lagrangian (comoving) coordinates in Minkowski space-time by means of an action principle whose Lagrangian is determined by the equation of state of the fluid. This action was reformulated as a parametrized Minkowski theory in Ref.[10] and then studied in the rest-frame instant form of dynamics. The main drawback of this approach is that we can get an explicit closed form of the Hamiltonian quantities only for few physically relevant equations of state, including the dust and the photon gas, due to the necessity of solving a trascendental equation (see Appendix B).

As a consequence, in this paper we studied the coupling of the dust to ADM tetrad gravity in globally hyperbolic, asymptotically Minkowskian space-times. We found the Hamiltonian formulation of the gravitational field with dynamical (not test) dust as matter and we gave the Hamilton equations of dust in the York canonical basis of Refs.[1, 2]. In this way we can disentangle the inertial gauge effects of the gravitational field from the tidal ones (the gravitational waves in the linearized theory). Also the Hamiltonian Post-Minkowskian linearization [3, 4], avoiding the Post-Newtonian expansion, is given in the 3-orthogonal Schwinger time gauges.

By using radar coordinates adapted to a time-like observer we define the Hamiltonian theory in global non-inertial frames, centered on the observer, with well-defined instantaneous 3-spaces, dynamically determined by Einstein's equations. In the York canonical basis the basic inertial canonical gauge variables are: i) three angles describing the freedom in the choice of the 3-coordinates inside the 3-spaces (their gauge fixing determines the shift functions); ii) the York time (the trace of the extrinsic curvature of the 3-spaces) describing the general relativistic remnant of the freedom in clock synchronization (its gauge fixing determines the lapse function).

In this framework we have studied the problem of selecting the subset of the irrotational motions of the dust. For this subset there are special 3-spaces determined by the dust 4-velocity and we have studied the problem of which gauge fixing is needed for having these dust 3-spaces coinciding with the 3-spaces of the global non-inertial frame. Also the Eulerian point of view [11] was discussed in the case of dust.

Since dust is the type of matter used in cosmological models (usually irrotational dust in comoving coordinates) and since there are indications that at least part of *dark matter* can be explained as a relativistic inertial effect induced by the York time [4, 20], the material of this paper is preparatory for the ADM formulation of cosmology. In particular we want to use this Hamiltonian description of dust in the framework of *back-reaction* [7, 21], in which an averaging procedure inside the 3-spaces (leading to a breaking of homogeneity and isotropy) opens the possibility to describe *dark energy*, and all the associated effects as the accelerated expansion of the universe, as an effect induced by the non-linearities of Einstein's equations. Since the spatial average is well defined only for 3-scalar functions and since most of the quantities describing the gravitational field and the dust are 3-scalars in the York canonical basis, we now have a framework for studying the spatial average of most of the Hamilton equations. Moreover we can study which cosmological notions have an inertial origin being functionals of the inertial gauge variable York time. These problems will be faced in a future paper.

Moreover we have to study the photon gas in the framework of this paper. Also we can look for Hamiltonian expressions of the fluid mass density, whose Lagrangian expression gives approximations to the equations of state usually used for fluids (an inverse procedure to the one used in Appendix B). This would open the possibility of studying *compact fluid bodies* without symmetries (relevant for the description of star) and their multipoles [22, 23] at the Hamiltonian level and to see whether our approach can be useful in formulating a well-posed Cauchy problem with a free boundary for the body, a still unsolved problem at the mathematical level [24].

Appendix A: Canonical ADM Tetrad Gravity and of the York Canonical Basis

In this Appendix we review the formulation of canonical ADM tetrad gravity developed In Refs.[1] starting from the ADM action considered as a functional of the cotetrads defined in Eqs.(2.1). Then we introduce the York canonical basis and the expansion and the shear of the congruence of the Eulerian observers.

1. The Original Canonical Variables of ADM Tetrad Gravity

As said in ref.[1–3] and with the notations of Eqs. (2.1)-(2.5), in ADM canonical tetrad gravity the 16 configuration variables are: the 3 boost variables $\varphi_{(a)}(\tau,\vec{\sigma})$; the lapse and shift functions $n(\tau,\vec{\sigma})$ and $n_{(a)}(\tau,\vec{\sigma})$; the cotriads ${}^3e_{(a)r}(\tau,\vec{\sigma})$. Their conjugate momenta are $\pi_{\varphi_{(a)}}(\tau,\vec{\sigma})$, $\pi_n(\tau,\vec{\sigma})$, $\pi_{n_{(a)}}(\tau,\vec{\sigma})$, ${}^3\pi^r_{(a)}(\tau,\vec{\sigma})$. There are 14 first-class constraints: A) the 10 primary constraints of Eqs.(4.4): $\pi_{\varphi_{(a)}}(\tau,\vec{\sigma}) \approx 0$, $\pi_n(\tau,\vec{\sigma}) \approx 0$, $\pi_{n_{(a)}}(\tau,\vec{\sigma}) \approx 0$ and the 3 rotation constraints $M_{(a)}(\tau,\vec{\sigma}) \approx 0$ implying the gauge nature of the 3 Euler angles $\alpha_{(a)}(\tau,\vec{\sigma})$; B) the 4 secondary super-Hamiltonian and super-momentum constraints $\mathcal{H}(\tau,\vec{\sigma}) \approx 0$, $\mathcal{H}_{(a)}(\tau,\vec{\sigma}) \approx 0$ of Eqs.(4.5). As a consequence there are 14 gauge variables (the *inertial effects*) and two pairs of canonically conjugate physical degrees of freedom (the *tidal effects*, which become the gravitational waves in the linearized theory).

The basis of canonical variables for this formulation of tetrad gravity, naturally adapted to 7 of the 14 first-class constraints, is (see Refs, [1, 25] for the direction-independent boundary conditions at spatial infinity)

$\varphi_{(a)}$	n	$n_{(a)}$	${}^3e_{(a)r}$
$\pi_{\varphi_{(a)}} \approx 0$	$\pi_n \approx 0$	$\pi_{n_{(a)}} \approx 0$	${}^3\pi^r_{(a)}$

$$
\begin{aligned}
&\{n(\tau,\vec{\sigma}), \pi_n(\tau,\vec{\sigma}^{'})\} = \delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\
&\{n_{(a)}(\tau,\vec{\sigma}), \pi_{n_{(b)}}(\tau,\vec{\sigma}^{'})\} = \delta_{(a)(b)}\delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\
&\{\varphi_{(a)}(\tau,\vec{\sigma}), \pi_{\varphi_{(b)}}(\tau,\vec{\sigma}^{'})\} = \delta_{(a)(b)}\delta^3(\vec{\sigma},\vec{\sigma}^{'}),\\
&\{{}^3e_{(a)r}(\tau,\vec{\sigma}), {}^3\pi^s_{(b)}(\tau,\vec{\sigma}^{'})\} = \delta_{(a)(b)}\delta^s_r\delta^3(\vec{\sigma},\vec{\sigma}^{'}).
\end{aligned}
\tag{A1}
$$

2. The York Canonical Basis

In Ref.[1] we studied a point canonical transformation on the canonical variables (A1), implementing the York map and adapted to 10 primary first-class constraints (it is a Shanmugadhasan canonical transformation adapted also to the constraints $M_{(a)}(\tau,\vec{\sigma}) \approx 0$). It is based on the fact that the 3-metric ${}^3g_{rs}$ is a real symmetric 3×3 matrix, which may be diagonalized with an *orthogonal* matrix $V(\theta^r)$, $V^{-1} = V^T$ ($\sum_u V_{ua}\, V_{ub} = \delta_{ab}$, $\sum_a V_{ua}\, V_{va} = \delta_{uv}$, $\sum_{uv} \epsilon_{wuv}\, V_{ua}\, V_{vb} = \sum_c \epsilon_{abc}\, V_{cw}$), $det\, V = 1$, depending on 3 Euler angles θ^r [11]. The gauge

[11] Due to the positive signature of the 3-metric, we define the matrix V with the following indices: V_{ru}. Since the choice of Shanmugadhasan canonical bases breaks manifest covariance, we will use the notation $V_{ua} = \sum_v V_{uv}\, \delta_{v(a)}$ instead of $V_{u(a)}$. We use the following types of indices: $a = 1,2,3$ and $\bar{a} = 1,2$.

Euler angles θ^r give a description of the 3-coordinate systems on Σ_τ from a local point of view, because they give the orientation of the tangents to the 3 coordinate lines through each point (their conjugate momenta are determined by the super-momentum constraints). We only consider 3-metrics with 3 distinct positive eigenvalues. In this way we get the following York canonical basis

$$
\begin{array}{|l|l|l|l|}
\hline
\varphi_{(a)} & n & n_{(a)} & {}^3e_{(a)r} \\
\hline
\pi_{\varphi_{(a)}} \approx 0 & \pi_n \approx 0 & \pi_{n_{(a)}} \approx 0 & {}^3\pi^r_{(a)} \\
\hline
\end{array}
$$

$$
\longrightarrow
\begin{array}{|ll|ll|l|l|l|}
\hline
\varphi_{(a)} & \alpha_{(a)} & n & \bar{n}_{(a)} & \theta^r & \tilde{\phi} & R_{\bar{a}} \\
\hline
\pi_{\varphi_{(a)}} \approx 0 & \pi^{(\alpha)}_{(a)} \approx 0 & \pi_n \approx 0 & \pi_{\bar{n}_{(a)}} \approx 0 & \pi^{(\theta)}_r & \pi_{\tilde{\phi}} & \Pi_{\bar{a}} \\
\hline
\end{array}
\tag{A2}
$$

In the York canonical basis we have (from now on we will use V_{ra} for $V_{ra}(\theta^n)$ to simplify the notation; we use the following definitions: $n_{(a)} \stackrel{def}{=} \sum_b R_{(a)(b)}(\alpha_{(c)})\, \bar{n}_{(b)}$, ${}^3e_{(a)r} \stackrel{def}{=} \sum_b R_{(a)(b)}(\alpha_{(c)})\, {}^3\bar{e}_{(b)r}$, ${}^3e^r_{(a)} \stackrel{def}{=} \sum_b R_{(a)(b)}(\alpha_{(c)})\, {}^3\bar{e}^r_{(b)}$, where $R_{(a)(b)}(\alpha_{(c)})$ are rotation matrices, $R^T = R^{-1}$)

$$
\begin{aligned}
{}^4g_{\tau\tau} &= \epsilon \Big[(1+n)^2 - \sum_a \bar{n}^2_{(a)}\Big],\\
{}^4g_{\tau r} &= -\epsilon \sum_a \bar{n}_{(a)}\, {}^3\bar{e}_{(a)r} = -\epsilon\, \tilde{\phi}^{1/3} \sum_a Q_a\, V_{ra}\, \bar{n}_{(a)},\\
{}^4g_{rs} &= -\epsilon\, {}^3g_{rs} = -\epsilon\, \tilde{\phi}^{2/3} \sum_a Q_a^2\, V_{ra}\, V_{sa}, \qquad Q_a = e^{\sum_{\bar{a}}^{1,2} \gamma_{\bar{a}a} R_{\bar{a}}},\\
\tilde{\phi} &= \phi^6 = \sqrt{\gamma} = \sqrt{det\, {}^3g} = {}^3\bar{e}, \qquad {}^3\bar{e}_{(a)r} = \tilde{\phi}^{1/3}\, Q_a\, V_{ra}, \qquad {}^3\bar{e}^r_{(a)} = \tilde{\phi}^{-1/3}\, Q_a^{-1}\, V_{ra},
\end{aligned}
$$

$$
\begin{aligned}
{}^3\pi^r_{(a)} &= \sum_b R_{(a)(b)}(\alpha_{(e)})\, \bar{\pi}^r_{(b)},\\
{}^3\bar{\pi}^r_{(a)} &\approx \tilde{\phi}^{-1/3} \Big[V_{ra}\, Q_a^{-1}\, (\tilde{\phi}\, \pi_{\tilde{\phi}} + \sum_{\bar{b}} \gamma_{\bar{b}a}\, \Pi_{\bar{b}}) +\\
&+ \sum_l^{l\neq a} \sum_{twi} Q_l^{-1}\, \frac{V_{rl}\, \epsilon_{alt}\, V_{wt}}{Q_l\, Q_a^{-1} - Q_a\, Q_l^{-1}}\, B_{iw}\, \pi^{(\theta)}_i\Big],
\end{aligned}
$$

$$
\begin{aligned}
\pi^{(\theta)}_i &= -\sum_{lmra} A_{ml}(\theta^n)\, \epsilon_{mir}\, {}^3e_{(a)l}\, {}^3\bar{\pi}^r_{(a)},\\
\pi_{\tilde{\phi}} &= \frac{c^3}{12\pi\, G}\, {}^3K \approx \frac{1}{3\; {}^3e} \sum_{ra} {}^3\bar{\pi}^r_{(a)}\, {}^3\bar{e}_{(a)r},\\
\Pi_{\bar{a}} &= \sum_{ra} \gamma_{\bar{a}a}\, {}^3\bar{\pi}^r_{(a)}\, {}^3\bar{e}_{(a)r}.
\end{aligned}
\tag{A3}
$$

The set of numerical parameters $\gamma_{\bar{a}a}$ satisfies [1, 13] $\sum_u \gamma_{\bar{a}u} = 0$, $\sum_u \gamma_{\bar{a}u}\gamma_{\bar{b}u} = \delta_{\bar{a}\bar{b}}$, $\sum_{\bar{a}} \gamma_{\bar{a}u}\gamma_{\bar{a}v} = \delta_{uv} - \frac{1}{3}$. Each solution of these equations defines a different York canonical basis. See Ref.[1] for the SO(3) Cartan matrices $A_{ij}(\theta^n)$ and $B = A^{-1}$.

In Eq.(A3) the quantity ${}^3K(\tau, \vec{\sigma})$ is the trace of the extrinsic curvature ${}^3K_{rs}(\tau, \vec{\sigma})$ of the instantaneous 3-spaces Σ_τ, whose expression in the York canonical basis is

$$
\begin{aligned}
{}^3K_{rs} \approx& -\frac{4\pi\, G}{c^3}\, \tilde{\phi}^{-1/3} \Big(\sum_a Q_a^2\, V_{ra}\, V_{sa}\, [2 \sum_{\bar{b}} \gamma_{\bar{b}a}\, \Pi_{\bar{b}} - \tilde{\phi}\, \pi_{\tilde{\phi}}] + \\
&+ \sum_{ab} Q_a\, Q_b\, (V_{ra}\, V_{sb} + V_{rb}\, V_{sa}) \sum_{twi} \frac{\epsilon_{abt}\, V_{wt}\, B_{iw}\, \pi_i^{(\theta)}}{Q_b\, Q_a^{-1} - Q_a\, Q_b^{-1}} \Big). \qquad \text{(A4)}
\end{aligned}
$$

The *York internal extrinsic time* ${}^3K(\tau, \vec{\sigma})$ is the only gauge variable among the momenta: this is a reflex of the Lorentz signature of space-time, because $\pi_{\tilde{\phi}}$ and θ^n can be used as a set of 4-coordinates [1]. Its conjugate variable, to be determined by the super-hamiltonian constraint, is $\tilde{\phi} = \phi^6 = {}^3\bar{e}$, which is proportional to *Misner's internal intrinsic time*; moreover $\tilde{\phi}$ is the *volume density* on Σ_τ: $V_R = \int_R d^3\sigma\, \phi^6$, $R \subset \Sigma_\tau$. Since we have ${}^3g_{rs} = \tilde{\phi}^{2/3}\, {}^3\hat{g}_{rs}$ with $det\, {}^3\hat{g}_{rs} = 1$, $\tilde{\phi}$ is also called the conformal factor of the 3-metric. The two pairs of 3-scalar canonical variables $R_{\bar{a}}$, $\Pi_{\bar{a}}$, $\bar{a} = 1, 2$, describe the generalized *tidal effects*, namely the independent degrees of freedom of the gravitational field. In particular the configuration tidal variables $R_{\bar{a}}$ depend *only on the eigenvalues of the 3-metric*. They are Dirac observables *only* with respect to the Hamiltonian gauge transformations generated by 10 of the 14 first class constraints.

Since the variables $\tilde{\phi}$ and $\pi_i^{(\theta)}$ are determined by the super-Hamiltonian and supermomentum constraints, the *arbitrary gauge variables* are $\alpha_{(a)}$, $\varphi_{(a)}$, θ^i, $\pi_{\tilde{\phi}}$, n and $\bar{n}_{(a)}$. As shown in Refs.[1], they describe the following generalized *inertial effects*:

a) $\alpha_{(a)}(\tau, \vec{\sigma})$ and $\varphi_{(a)}(\tau, \vec{\sigma})$ are the 6 configuration variables parametrizing the O(3,1) gauge freedom in the choice of the tetrads in the tangent plane to each point of Σ_τ and describe the arbitrariness in the choice of a tetrad to be associated to a time-like observer, whose world-line goes through the point $(\tau, \vec{\sigma})$. They fix *the unit 4-velocity of the observer and the conventions for the orientation of gyroscopes and their transport along the world-line of the observer*.

b) $\theta^i(\tau, \vec{\sigma})$ describe the arbitrariness in the choice of the 3-coordinates in the instantaneous 3-spaces Σ_τ of the chosen non-inertial frame centered on an arbitrary time-like observer. Their choice will induce a pattern of *relativistic inertial forces* for the gravitational field , whose potentials are the functions $V_{ra}(\theta^i)$ present in the weak ADM energy E_{ADM} [1, 2].

c) $\bar{n}_{(a)}(\tau, \vec{\sigma})$, the shift functions appearing in the Dirac Hamiltonian, describe which points on different instantaneous 3-spaces have the same numerical value of the 3-coordinates. They are the inertial potentials describing the effects of the non-vanishing off-diagonal components ${}^4g_{\tau r}(\tau, \vec{\sigma})$ of the 4-metric, namely they are the *gravito-magnetic potentials* responsible of effects like the dragging of inertial frames (Lens-Thirring effect) in the post-Newtonian approximation. The shift functions are determined by the τ-preservation of the gauge fixings determining the gauge variables $\theta^i(\tau, \vec{\sigma})$.

d) $\pi_{\tilde{\phi}}(\tau, \vec{\sigma})$, i.e. the York time ${}^3K(\tau, \vec{\sigma})$, describes the non-dynamical arbitrariness in the choice of the convention for the synchronization of distant clocks which remains in the transition from special to general relativity. The York time (being a momentum) gives rise to a *semi-definite negative kinetic term* in the Dirac Hamiltonian [1–3]. This *unusual inertial effect* is connected to the problem of the relativistic non-dynamical freedom in the choice of the *instantaneous 3-space*, which has no non-relativistic analogue (in Galilei space-time time is absolute and there is an absolute notion of Euclidean 3-space). Its effects are completely unexplored.

e) $n(\tau, \vec{\sigma})$, the lapse function appearing in the Dirac Hamiltonian, describes the arbitrariness in the choice of the unit of proper time in each point of the simultaneity surfaces Σ_τ, namely how these surfaces are packed in the 3+1 splitting. The lapse function is determined by the τ-preservation of the gauge fixing for the gauge variable ${}^3K(\tau, \vec{\sigma})$.

3. The Expansion and the Shear of the Eulerian Observers.

Let us now consider the geometrical interpretation of the extrinsic curvature ${}^3K_{rs}$ of the instantaneous 3-spaces Σ_τ in terms of the properties of the surface-forming (i.e. irrotational) congruence of Eulerian (non geodesic) time-like observers, whose world-lines have the tangent unit 4-velocity equal to the unit normal orthogonal to the instantaneous 3-spaces Σ_τ. If we use radar 4-coordinates, the covariant unit normal $\epsilon\, l_A = (1+n)\,(1;0)$ of Eqs.(2.4) has the following covariant derivative

$$
\begin{aligned}
{}^4\nabla_A\, \epsilon\, l_B \;&=\; \epsilon\, l_A\, {}^3a_B + \sigma_{AB} + \frac{1}{3}\,\theta\, h_{AB} - \omega_{AB} = \epsilon\, l_A\, {}^3a_B + {}^3K_{AB},\\
{}^3K_{AB} \;&=\; {}^3K_{rs}\, \hat{b}^r_A\, \hat{b}^s_B, \qquad \hat{b}^r_A = \delta^r_A + {}^3\bar{e}^r_{(a)}\, \bar{n}_{(a)}\, \delta^\tau_A, \qquad h_{AB} = {}^4g_{AB} - \epsilon\, l_A\, l_B.
\end{aligned}
\tag{A5}
$$

The quantities appearing in Eqs.(A5) are:

a) the *acceleration* of the Eulerian observers

$$
\begin{aligned}
{}^3a^A \;&=\; l^B\, {}^4\nabla_B\, l^A = {}^4g^{AB}\, {}^3a_B, \qquad {}^3a_A = {}^3a_r\, \hat{b}^r_A,\\
&{}^3a_r = \partial_r\, ln\,(1+n) = {}^3a_r\, {}^3\bar{e}^r_{(a)}\, {}^3\bar{e}^u_{(a)}\, n_u = {}^3a_r\, {}^3\bar{e}^r_{(a)}\, \bar{n}_{(a)},\\
&{}^3a_r = -\epsilon\, {}^3\bar{e}^r_{(a)}\, {}^3\bar{e}^s_{(a)}\, {}^3a_s, \qquad {}^3a^\tau = 0;
\end{aligned}
\tag{A6}
$$

b) the *vorticity* or *twist* (a measure of the rotation of the nearby world-lines infinitesimally surrounding the given one), which is vanishing because the congruence is surface-forming

$$
\begin{aligned}
\omega_{AB} \;&=\; -\omega_{BA} = \frac{\epsilon}{2}\,(l_A\, {}^3a_B - l_B\, {}^3a_A) - \frac{\epsilon}{2}\,({}^4\nabla_A\, l_B - {}^4\nabla_B\, l_A) = 0,\\
&\omega_{AB}\, l^B = 0, \qquad \omega^A = \frac{1\epsilon}{2}\, {}^4\eta^{ABCD}\, \omega_{BC}\, l_D = 0;
\end{aligned}
\tag{A7}
$$

c) the *expansion* [12], which coincides with the *York external time*, is proportional in cosmology to the *Hubble parameter* H [13] and determines the dimensionless (cosmological) *deceleration parameter* $q = 3\, l^A\, {}^4\nabla_A\, \frac{1}{\theta} - 1 = -3\, \theta^{-2}\, l^A\, \partial_A\, \theta - 1$,

$$\begin{aligned}
\theta &= {}^4\nabla_A\, l^A = -\epsilon\, {}^3K = -\frac{4\pi\, G}{c^3}\, \frac{{}^3\bar{e}_{(a)r}\, {}^3\bar{\pi}^r_{(a)}}{{}^3\bar{e}} = -\epsilon\, \frac{12\pi\, G}{c^3}\, \pi_{\tilde{\phi}},\\
H &= \frac{1}{3}\, \theta = \frac{1}{l}\, l^A\, {}^4\nabla_A\, l = -\epsilon\, \frac{4\pi\, G}{c^3}\, \pi_{\tilde{\phi}}, \qquad q = 3\, l^A\, {}^4\nabla_A\, \frac{1}{\theta} - 1; \qquad \text{(A8)}
\end{aligned}$$

d) the *shear* [14]

$$\begin{aligned}
\sigma_{AB} &= \sigma_{BA} = -\frac{\epsilon}{2}\, ({}^3a_A\, l_B + {}^3a_B\, l_A) + \frac{\epsilon}{2}\, ({}^4\nabla_A\, l_B + {}^4\nabla_B\, l_A) - \frac{1}{3}\, \theta\, {}^3h_{AB} =\\
&= ({}^3K_{rs} - \frac{1}{3}\, {}^3g_{rs}\, {}^3K)\, \hat{\bar{b}}^r_A\, \hat{\bar{b}}^s_B, \qquad {}^4g^{AB}\, \sigma_{AB} = 0, \qquad \sigma_{AB}\, l^B = 0. \qquad \text{(A9)}
\end{aligned}$$

By explicit calculation we get the following components of the shear along the tetrads (2.2)

$$\begin{aligned}
\sigma_{AB} &= \sigma_{(\alpha)(\beta)}\, {}^4\mathring{\bar{E}}^{(\alpha)}_A\, {}^4\mathring{\bar{E}}^{(\beta)}_B = {}^4g_{AC}\, {}^4g_{BD}\, \sigma^{CD}, \qquad \sigma_{(\alpha)(\beta)} = \sigma_{AB}\, {}^4\mathring{\bar{E}}^A_{(\alpha)}\, {}^4\mathring{\bar{E}}^B_{(\beta)},\\
\sigma_{(o)(o)} &= 0, \qquad \sigma_{(o)(a)} = 0,\\
\sigma_{(a)(b)} &= \sigma_{(b)(a)} = ({}^3K_{rs} - \frac{1}{3}\, {}^3g_{rs}\, {}^3K)\, {}^3\bar{e}^r_{(a)}\, {}^3\bar{e}^s_{(b)}, \qquad \sum_a \sigma_{(a)(a)} = 0. \qquad \text{(A10)}
\end{aligned}$$

$\sigma_{(a)(b)}$ depends upon θ^r, $\tilde{\phi}$, $R_{\bar{a}}$, $\pi^{(\theta)}_r$ and $\Pi_{\bar{a}}$.

As a consequence, by using Eqs.(A3) and (A4) we have

---

[12] It measures the average expansion of the infinitesimally nearby world-lines surrounding a given world-line in the congruence.

[13] l is a representative length along the integral curves of ${}^4\mathring{\bar{E}}^A_{(o)}$, describing the volume expansion (contraction) behavior of the congruence.

[14] It measures how an initial sphere in the tangent space to the given world-line, which is Lie-transported along the world-line tangent l^μ (i.e. it has zero Lie derivative with respect to $l^\mu\, \partial_\mu$), is distorted towards an ellipsoid with principal axes given by the eigenvectors of $\sigma^\mu{}_\nu$, with rate given by the eigenvalues of $\sigma^\mu{}_\nu$.

$$
\begin{aligned}
\tilde{\phi}\,\sigma_{(a)(a)} &= -\frac{8\pi\,G}{c^3}\sum_{\bar{a}}\gamma_{\bar{a}a}\,\Pi_{\bar{a}}, \rightarrow\; \Pi_{\bar{a}} = -\frac{c^3}{8\pi\,G}\,\tilde{\phi}\sum_a \gamma_{\bar{a}a}\,\sigma_{(a)(a)},\\
\tilde{\phi}\,\sigma_{(a)(b)}|_{a\neq b} &= -\frac{8\pi\,G}{c^3}\sum_{tw}\frac{\epsilon_{abt}\,V_{wt}}{Q_b\,Q_a^{-1}-Q_a\,Q_b^{-1}}\sum_i B_{iw}\,\pi_i^{(\theta)},\\
\Rightarrow \qquad & \pi_i^{(\theta)} = \frac{c^3}{8\pi\,G}\,\tilde{\phi}\sum_{wtab} A_{wi}\,V_{wt}\,Q_a\,Q_b^{-1}\,\epsilon_{tab}\,\sigma_{(a)(b)}|_{a\neq b},\\
{}^3K_{rs} &= -\frac{\epsilon}{3}\,{}^3g_{rs}\,\theta + \sigma_{(a)(b)}\,{}^3\bar{e}_{(a)r}\,{}^3\bar{e}_{(b)s}.
\end{aligned}
\tag{A11}
$$

Therefore the diagonal elements of the shear of the Eulerian observers describe the tidal momenta $\Pi_{\bar{a}}$, while the non-diagonal elements determine the variables $\pi_i^{(\theta)}$, determined by the super-momentum constraints. Moreover their expansion θ is the inertial gauge variable determining the non-dynamical part (general relativistic gauge freedom in clock synchronization) of the shape of the instantaneous 3-spaces Σ_τ.

Appendix B: The Fluid Velocity in terms of the Fluid Momentum

Let us restrict the action (3.4) to isentropic, $s = const.$, perfect fluids. As shown in Ref.[10] the Lagrangian and the fluid momenta have the form [15]

$$\begin{aligned}
\tilde{n} &= \frac{|J|}{(1+n)\sqrt{\gamma}} \stackrel{def}{=} \frac{X}{\sqrt{\gamma}}, \qquad \rho = \rho(\tilde{n}) = \rho(\frac{|J|}{(1+n)\sqrt{\gamma}}) \stackrel{def}{=} \mu\, f(\frac{X}{\sqrt{\gamma}}),\\
\Rightarrow &\quad L = -\mu\,(1+n)\sqrt{\gamma} f(\frac{X}{\sqrt{\gamma}}),\\
\Rightarrow &\quad \Pi_i(\tau,\vec{\sigma}) = \frac{\partial L}{\partial\, \partial_\tau\, \alpha^i}(\tau,\vec{\sigma}) = -\mu\Big[\frac{\partial f(x)}{\partial x}|_{x=\frac{X}{\sqrt{\gamma}}} \frac{Y^r\, \mathcal{T}_{ri}}{X}\Big](\tau,\vec{\sigma}).
\end{aligned} \tag{B1}$$

Eq.(3.7) becomes

$$\begin{aligned}
&X^2\Big[\mu^2\,(J^\tau)^2 + A^2(\frac{\partial f(x)}{\partial x}|_{x=\frac{X}{\sqrt{\gamma}}})^{-2}\Big] = \mu^2 (J^\tau)^4, \qquad A^2 = \sum_{rsij} {}^3g^{rs}\, \partial_r\, \alpha^i\, \Pi_i\, \partial_s\, \alpha^j\, \Pi_j,\\
\Rightarrow &\quad X = \sqrt{\gamma}\,\tilde{n} = F(\sqrt{\gamma}, A^2, (J^\tau)^2)[\rho],\\
\stackrel{(3.11)}{\Rightarrow} &\quad \mathcal{M} = \frac{A^2\, X}{\mu\,(J^\tau)^2\, \frac{\partial\, f(x)}{\partial\, x}|_{x=\frac{X}{\sqrt{\gamma}}}} + \mu\,\sqrt{\gamma}\, f(\frac{X}{\sqrt{\gamma}}).
\end{aligned} \tag{B2}$$

Some possible equations of state for barotropic fluids, i.e. with $p = p(\rho)$, are (in the isentropic case one gets $\rho = \rho(\tilde{n})$ by solving $p(\rho(\tilde{n})) = \tilde{n}\frac{\partial \rho(\tilde{n})}{\partial \tilde{n}} - \rho(\tilde{n})$; the definition of the sound velocity is $v_s^2 = c^2\, \frac{\partial\, p(\rho)}{\partial\, \rho}$):

1) $p = 0$, dust: this implies

$$\rho(\tilde{n}) \;=\; \mu\,\tilde{n} = \mu\frac{X}{\sqrt{\gamma}}, \qquad i.e. \qquad f(\frac{X}{\sqrt{\gamma}}) = \frac{X}{\sqrt{\gamma}}, \qquad \frac{\partial f(\frac{X}{\sqrt{\gamma}})}{\partial X} = \frac{1}{\sqrt{\gamma}}. \tag{B3}$$

The equation for X and its solution are

[15] In general μ is not the chemical potential but only a parameter. The quantities X, Y^r and $\mathcal{T}_{ti}$ are defined in Eq.(3.6). We have $\frac{\partial X}{\partial N}|_{N=1+n} = \frac{\sum_{rs} {}^3g_{rs}\, Y^r\, Y^s}{(1+n)\, X} = \frac{(J^\tau)^2 - X^2}{(1+n)\, X}$, $\frac{\partial X}{\partial n^u} = -J^\tau \frac{\sum_s {}^3g_{us}\, Y^s}{(1+n)\, X}$.

$$
\begin{aligned}
X^2 \quad & [\mu^2\,(J^\tau)^2 + A^2] = B^2,\\
X \;&=\; \frac{\mu (J^\tau)^2}{\sqrt{\mu^2\,(J^\tau)^2 + \sum_{rsij}\,{}^3g^{rs}\,\partial_r\,\alpha^i\,\partial_s\,\alpha^j\,\Pi_i\,\Pi_j}},\\
Y^r \;&=\; -\frac{|J^\tau|\,\sum_{si}\,{}^3g^{rs}\,\partial_s\,\alpha^i\,\Pi_i}{\sqrt{\mu^2\,(J^\tau)^2 + \sum_{uvij}\,{}^3g^{uv}\,\partial_u\,\alpha^i\,\partial_v\,\alpha^j\,\Pi_i\,\Pi_j}},\\
\Rightarrow \quad & \mathcal{M} = \sqrt{\mu^2\,(J^\tau)^2 + \sum_{rsij}\,{}^3g^{rs}\,\partial_r\,\alpha^i\,\partial_s\,\alpha^j\,\Pi_i\,\Pi_j}.
\end{aligned}
\tag{B4}
$$

2) $p = k\rho(\tilde{n}) = \tilde{n}\frac{\partial \rho(\tilde{n})}{\partial \tilde{n}} - \rho(\tilde{n})$ ($k \neq -1$ because otherwise $\rho = const.$, $\mu = 0$). The previous differential equation for $\rho(\tilde{n})$ implies

$$
\begin{aligned}
\rho(\tilde{n}) \;&=\; \mu \tilde{n}^{k+1} = \mu(\frac{X}{\sqrt{\gamma}})^{k+1}, \qquad i.e.\\
f(\frac{X}{\sqrt{\gamma}}) \;&=\; (\frac{X}{\sqrt{\gamma}})^{k+1}, \qquad \frac{\partial f(\frac{X}{\sqrt{\gamma}})}{\partial X} = \frac{k+1}{\sqrt{\gamma}}(\frac{X}{\sqrt{\gamma}})^k,
\end{aligned}
\tag{B5}
$$

[for $k \to 0$ we recover case 1)]. More in general one can have $k = k(s)$: this is a non-isentropic perfect fluid with $\rho = \rho(\tilde{n}, s)$.

The equation for X is

$$
X^2\,[\mu^2\,(J^\tau)^2 + \frac{A^2}{(k+1)^2(\frac{X}{\sqrt{\gamma}})^{2k}}] = B^2.
\tag{B6}
$$

In general this equation cannot be solved explicitly, but is soluble for $k = \frac{1}{3}, \frac{1}{2}, 1, 2, -\frac{1}{3}, -\frac{1}{2}, -2$: the equation for X is linear for $k = 1$, quadratic for $k = 2, \frac{1}{2}$, cubic for $k = \frac{1}{3}, -\frac{1}{2}, -2$ and biquadratic for $k = -\frac{1}{3}$.

By using Section V of Ref.[10] we get the following results for the cases $k = 1, 2, \frac{1}{2}, \frac{1}{3}$ (A^2 is defined in Eq.(A2)):

2a) $k = 1$, $p = \rho$, $\rho = \mu\,\tilde{n}^2$

$$
\begin{aligned}
X \;&=\; \frac{1}{2\,\mu\,J^\tau}\sqrt{\frac{4\,\mu^2}{\gamma}\,(J^\tau)^2 + A^2},\\
\mathcal{M} \;&=\; \frac{2\,\gamma^{-1}\,\mu^2\,(J^\tau)^2 - (\gamma + \frac{1}{2})\,A^2}{2\,\mu\,\sqrt{\gamma}\,(J^\tau)^2}.
\end{aligned}
\tag{B7}
$$

2b) $k = 2$, $p = 2\,\rho$, $\rho = \mu\,\tilde{n}^3$

$$X \;=\; \frac{J^\tau}{\sqrt{2}}\,\sqrt{1+\sqrt{1-\frac{4\,\gamma^2\,A^2}{9\,\mu^2\,(J^\tau)^6}}},$$
$$\mathcal{M} \;=\; \frac{\mu}{\gamma}\,\Big(X^2+\frac{\gamma^2\,A^2}{3\,\mu^2\,(J^\tau)^2\;X}\Big). \tag{B8}$$

2c) $k = \frac{1}{2}$, $p = \frac{1}{2}\,\rho$, $\rho = \mu\,\tilde{n}^{3/2}$:

$$X \;=\; \frac{1}{\mu}\,\Big(\sqrt{\mu^2\,(J^\tau)^2+\frac{\gamma\,A^4}{9\,\mu^2\,(J^\tau)^4}}-\frac{\gamma^{1/2}\,A^2}{3\,\mu\,(J^\tau)^2}\Big),$$
$$\mathcal{M} \;=\; \mu\,\gamma^{-1/4}\,\sqrt{X}\,\Big(X+\frac{2\,\gamma^{1/2}\,A^2}{3\,\mu^2\,(J^\tau)^2}\Big). \tag{B9}$$

2d) photon gas, $k = \frac{1}{3}$, $p = \frac{1}{3}\,\rho$, $\rho = \mu\,\tilde{n}^{4/3}$:

$$X \;=\; |J^\tau|\Big[-\frac{3\,A^2\,\gamma^{1/3}}{16\,\mu\,(J^\tau)^{8/3}}+\frac{1}{2^{1/3}}\,\Big(1-\frac{27\,A^6\,\gamma}{2^{11}\,\mu^6\,(J^\tau)^8}+\sqrt{1-\frac{27\,A^6\,\gamma}{2^{10}\,\mu^6\,(J^\tau)^8}}\;\Big)^{1/3}-$$
$$-\;\frac{1}{2^{1/3}}\,\Big(-1+\frac{27\,A^6\,\gamma}{2^{11}\,\mu^6\,(J^\tau)^8}+\sqrt{1-\frac{27\,A^6\,\gamma}{2^{10}\,\mu^6\,(J^\tau)^8}}\;\Big)^{1/3}\Big]^{3/2},$$
$$\mathcal{M} \;=\; \mu\,\gamma^{-1/6}\,X^{2/3}\,\Big(X^{2/3}+\frac{3\,\gamma^{1/3}\,a^2}{4\,\mu^2\,(J^\tau)^2}\Big). \tag{B10}$$

3) $p = k\rho^\gamma(\tilde{n}) = \tilde{n}\frac{\partial\rho(\tilde{n})}{\partial\tilde{n}} - \rho(\tilde{n})$ $(\gamma \neq 1)$ [26]. It is an isentropic polytropic perfect fluid $(\gamma = 1 + \frac{1}{n})$. The differential equation for $\rho(\tilde{n})$ implies [a is an integration constant; the chemical potential is $\mu = \frac{\partial\rho}{\partial\tilde{n}}|_s$]

$$\rho(\tilde{n}) \;=\; \frac{a\tilde{n}}{[1-k(a\tilde{n})^{\gamma-1}]^{\frac{1}{\gamma-1}}} = \frac{an}{[1-k(an)^{\frac{1}{n}}]^n}, \qquad i.e.$$
$$f(\frac{X}{\sqrt{\gamma}}) \;=\; \frac{\frac{X}{\sqrt{\gamma}}}{[1-k(a\frac{X}{\sqrt{\gamma}})^{\gamma-1}]^{\frac{1}{\gamma-1}}} = \frac{\frac{X}{\sqrt{\gamma}}}{[1-k(a\frac{X}{\sqrt{\gamma}})^{\frac{1}{n}}]^n},$$
$$\frac{\partial f(\frac{X}{\sqrt{\gamma}})}{\partial X} \;=\; \frac{1}{\sqrt{\gamma}}[1-k(a\frac{X}{\sqrt{\gamma}})^{\gamma-1}]^{-\frac{\gamma}{\gamma-1}} = \frac{1}{\sqrt{\gamma}}[1-k(a\frac{X}{\sqrt{\gamma}})^{\frac{1}{n}}]^{-(n+1)}. \tag{B11}$$

Instead in Ref.[27, 28] a polytropic perfect fluid is defined by the equation of state

$$\rho(\tilde{n}, \tilde{n}\, s) = \mu\, \tilde{n} + \frac{k(s)}{\gamma - 1}\, (\mu\, \tilde{n})^{\gamma}, \tag{B12}$$

and has pressure $p = k(s)(\mu\, \tilde{n})^{\gamma} = (\gamma - 1)(\rho - \mu\, \tilde{n})$. The (in general non explicitly soluble) equation for X is

$$X^2 \left[\mu^2\, (J^{\tau})^2 + A^2 \Big(1 - k(\mu \frac{X}{\sqrt{\gamma}})^{\frac{1}{n}}\Big)^{2(n+1)}\right] = B^2. \tag{B13}$$

In Ref. [10] there is also a discussion of the relativistic ideal non-isentropic (Boltzmann) gas [29] ($p = \tilde{n} k_B T$, $\rho = mc^2 \tilde{n} \Gamma(\beta) - p$, $\mu = \frac{\rho + p}{\tilde{n}} = mc^2 \Gamma(\beta)$).

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